Analysis of Algorithms
CS 477/677

Instructor: Monica Nicolescu

Lecture 23

Searching in a Graph

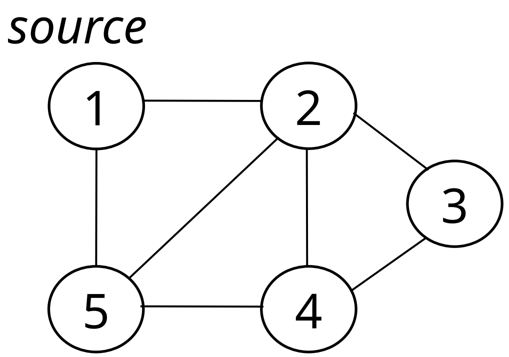
- **Graph searching** = systematically follow the edges of the graph so as to visit the vertices of the graph
- Two basic graph searching algorithms:
  - Breadth-first search
  - Depth-first search
- The difference between them is in the order in which they explore the unvisited edges of the graph
- Graph algorithms are typically elaborations of the basic graph-searching algorithms

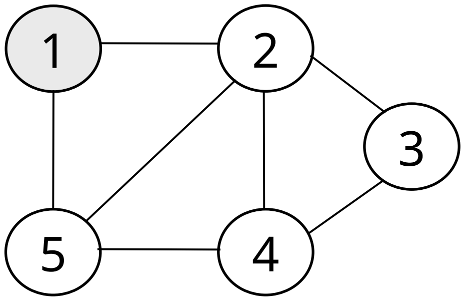
Breadth-First Search (BFS)

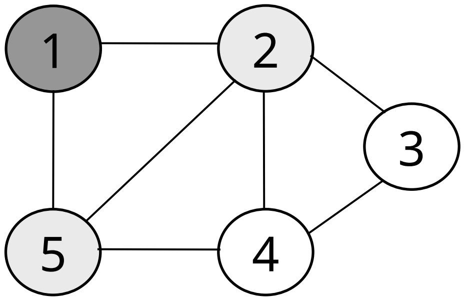
- **Input**:
  - A graph G = (V, E) (directed or undirected)
  - A **source** vertex s from V
- **Goal**:
  - Explore the edges of G to “discover” every vertex reachable from s, taking the ones closest to s first
- **Output**:
  - d[v] = distance (smallest # of edges) from s to v, for all v from V
  - A “breadth-first tree” rooted at s that contains all reachable vertices

Breadth-First Search (cont.)

- Keeping track of progress:
  - Color each vertex in either **white**, **gray** or **black**
  - Initially, all vertices are **white**
  - When being discovered a vertex becomes **gray**
  - After discovering all its adjacent vertices the node becomes **black**
  - Use FIFO queue *Q* to maintain the set of gray vertices

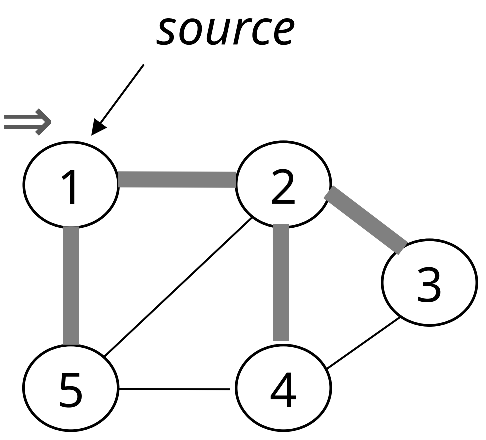






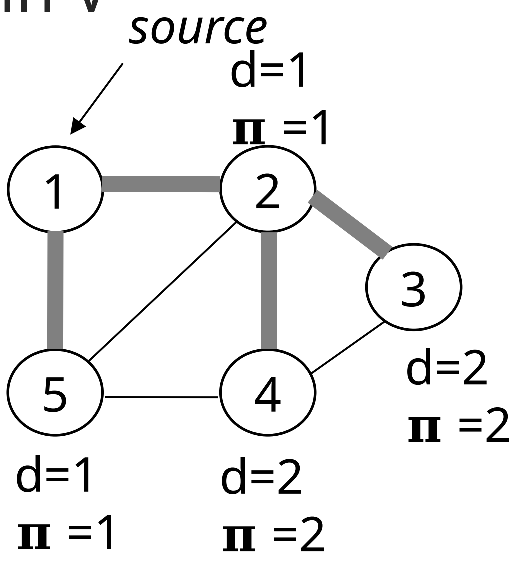
Breadth-First Tree

- BFS constructs a breadth-first tree
  - Initially contains the root (source vertex s)
  - When vertex v is discovered while scanning the adjacency list of a vertex u ⇒ vertex v and edge (u, v) are added to the tree
  - u is the **predecessor** (**parent**) of v in the breadth-first tree
  - A vertex is discovered only once ⇒ it has only one parent



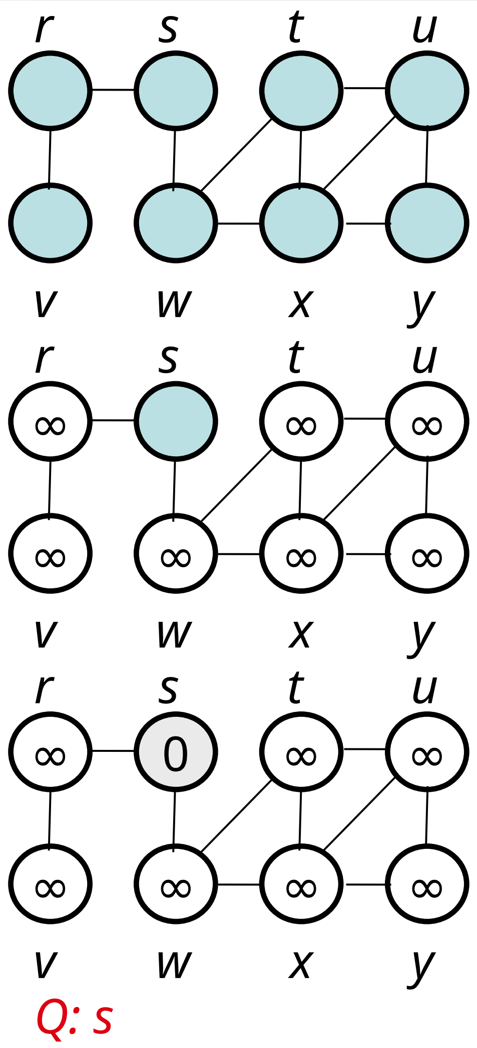
BFS Additional Data Structures

- G = (V, E) represented using adjacency lists
- color[u] – the color of the vertex for all u in V
- π[u] – predecessor of u
  - If u = s (root) or node u has not yet been discovered then [u] = π NIL
- d[u] – the distance from the source s to vertex u
- Use a FIFO queue *Q* to maintain the set of gray vertices



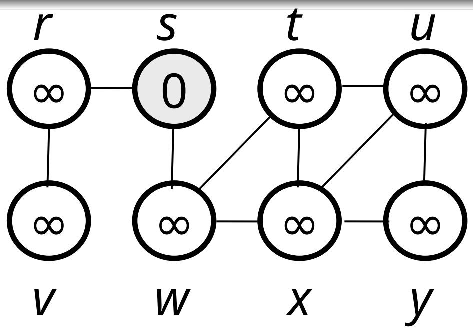
BFS(V, E, s)

```
1.  for each u in V - {s}
2.          do color[u] = WHITE
3.              d[u] ← ∞
4.              π[u] = NIL
5.  color[s] = GRAY
6.  d[s] ← 0
7.  π[s] = NIL
8.  Q = empty
9.  Q ← ENQUEUE(Q, s)
```

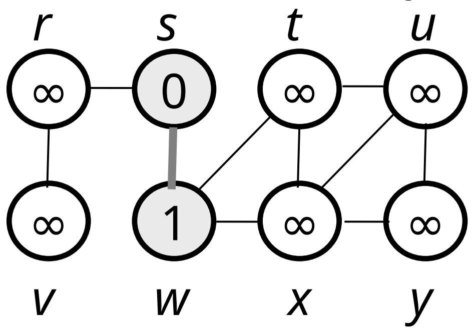


BFS(V, E, s)

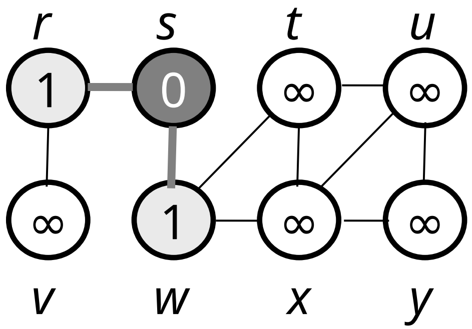
```
10. while Q not empty
11.     do u ← DEQUEUE(Q)
12.         for each v in Adj[u]
13.             do if color[v] = WHITE
14.                 then color[v] = GRAY
15.                     d[v] ← d[u] + 1
16.                         π[v] = u
17.                     ENQUEUE(Q, v)
18.     color[u] = BLACK
```



Q: s

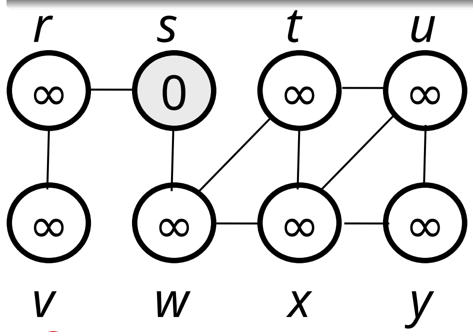


Q: w

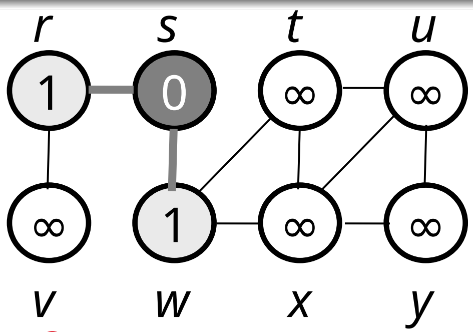


Q: w, r

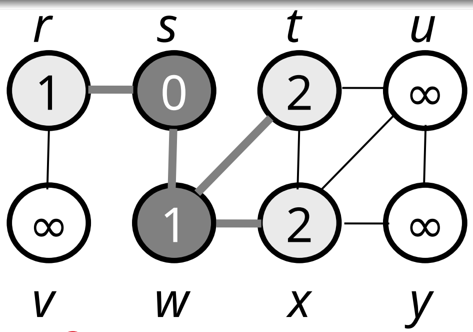
Example



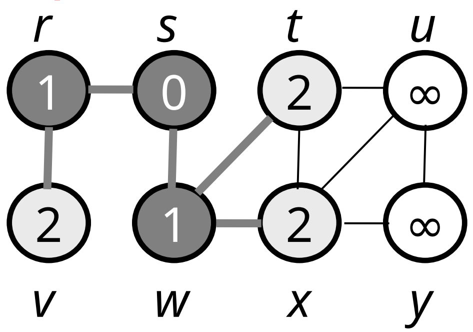
Q: s



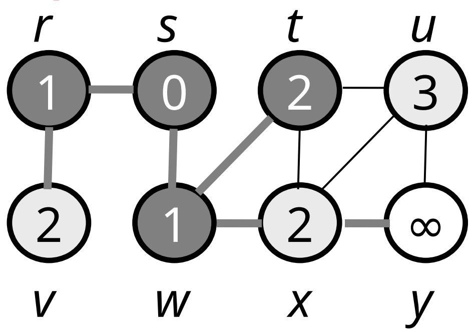
Q: w, r



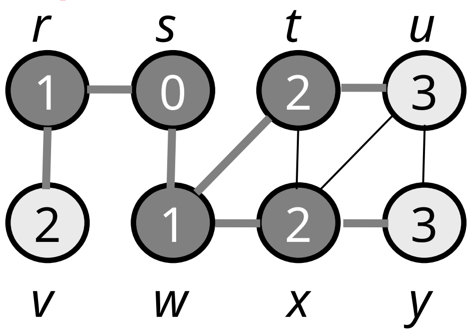
Q: r, t, x



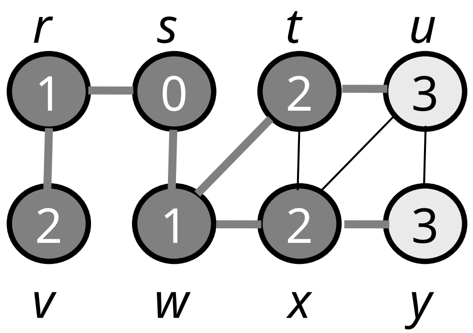
Q: t, x, v



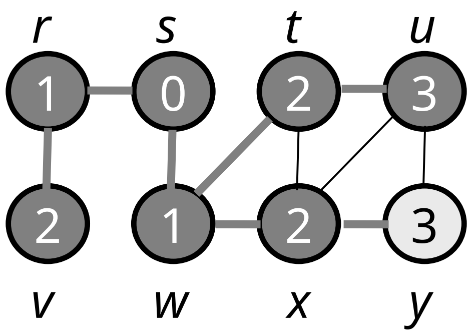
Q: x, v, u



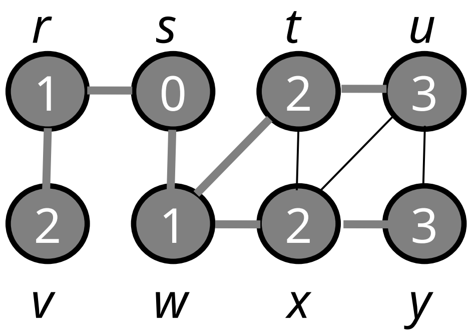
Q: v, u, y



Q: u, y



Q: y



Q: ∅

Analysis of BFS

```
1.  for each u ∈ V - {s}                  ⎫
2.      do color[u] ← WHITE               ⎬  O(|V|)
3.         d[u] ← ∞                       ⎭
4.         π[u] = NIL                     ⎫
5.  color[s] ← GRAY                       ⎪
6.  d[s] ← 0                              ⎬  Θ(1)
7.  π[s] = NIL                            ⎪
8.  Q ← ∅                                 ⎭
9.  Q ← ENQUEUE(Q, s)
```

Analysis of BFS

```
10. while Q not empty                         ←─────────────────┐
11.   do u ← DEQUEUE(Q)                       ←──── Θ(1)        │
12.     for each v in Adj[u]                  ←──── Scan Adj[u] for all vertices
13.       do if color[v] = WHITE                    u in the graph
14.         then color[v] = GRAY
15.           d[v] ← d[u] + 1
16.           π[v] = u
17.           ENQUEUE(Q, v)                   ←──── Θ(1)
18.     color[u] = BLACK
```

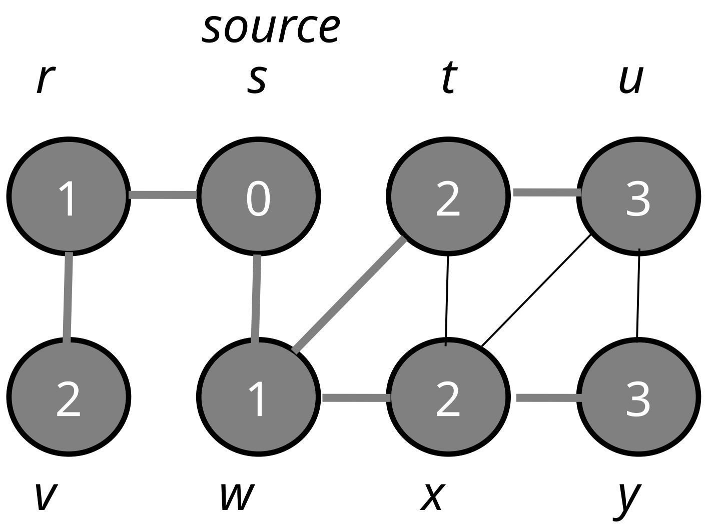
Scan Adj[u] for all vertices u in the graph

- Each vertex u is processed only once, when the vertex is dequeued
  - Sum of lengths of all adjacency lists = Θ(|E|)
  - Scanning operations: O(|E|)

- Total running time for BFS = O(|V| + |E|)

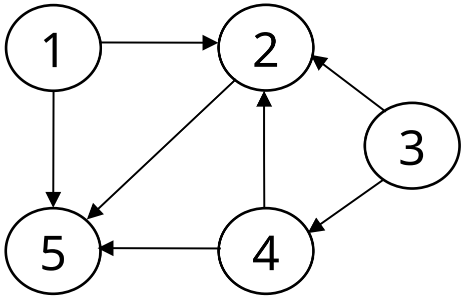
Shortest Paths Property

- BFS finds the shortest-path distance from the source vertex s ∈ V to each node in the graph
- Shortest-path distance = δ(s, u)
  - Minimum number of edges in any path from s to u

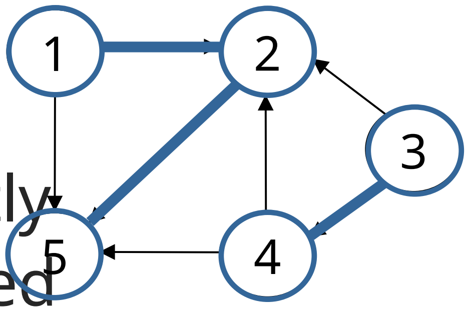


Depth-First Search

- **Input:**
  - G = (V, E) (No source vertex given!)
- **Goal**:
  - Explore the edges of G to “discover” every vertex in V starting at the most current visited node
  - Search may be repeated from multiple sources
- **Output:**
  - 2 **timestamps** on each vertex:
    - d[v] = discovery time
    - f[v] = finishing time (done with examining v’s adjacency list)
  - Depth-first forest

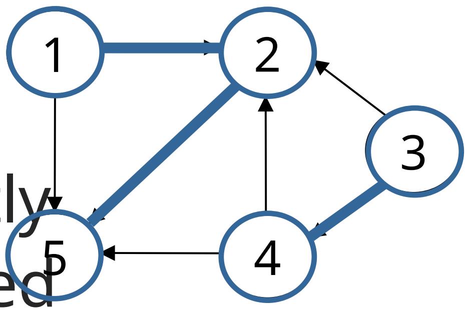


Depth-First Search



- Search "deeper" in the graph whenever possible
- Edges are explored out of the most recently discovered vertex v that still has unexplored edges
- After all edges of v have been explored, the search "backtracks" from the parent of v
- The process continues until all vertices reachable from the original source have been discovered
- If undiscovered vertices remain, choose one of them as a new source and repeat the search from that vertex
- DFS creates a "depth-first forest"

Depth-First Search

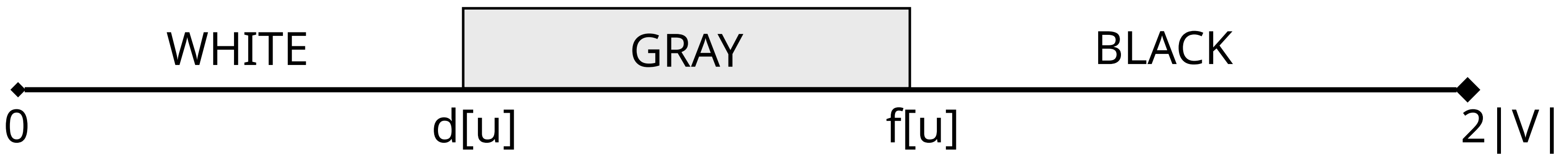


- Search "deeper" in the graph whenever possible
- Edges are explored out of the most recently discovered vertex v that still has unexplored edges
- After all edges of v have been explored, the search "backtracks" from the parent of v
- The process continues until all vertices reachable from the original source have been discovered
- If undiscovered vertices remain, choose one of them as a new source and repeat the search from that vertex
- DFS creates a "depth-first forest"

DFS Additional Data Structures

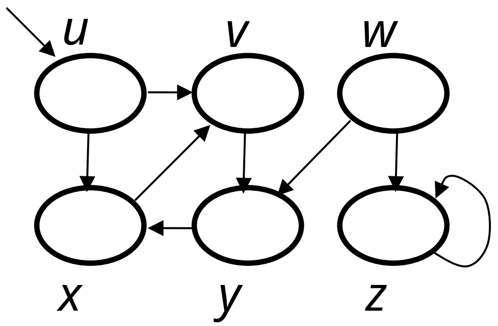
- Global variable: time-step
  - Incremented when nodes are discovered/finished
- color[u] – similar to BFS
  - White before discovery, gray while processing and black when finished processing
- π[u] – predecessor of u
- d[u], f[u] – discovery and finish times

$$1 \leq d[u] < f[u] \leq 2\,|V|$$



DFS(V, E)

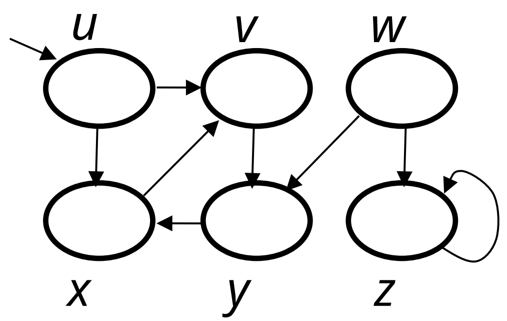
```
1. for each u ∈ V
2.      do color[u] ← WHITE
3.          π[u] ← NIL
4. time ← 0
5. for each u ∈ V
6.      do if color[u] = WHITE
7.              then DFS-VISIT(u)
```



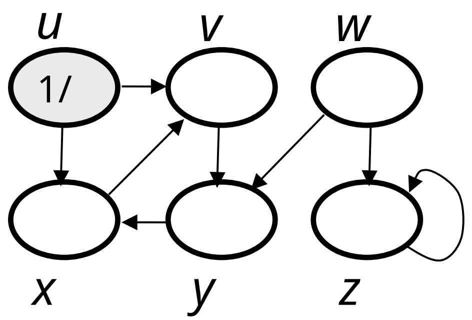
- Every time DFS-VISIT(u) is called, u becomes the root of a new tree in the depth-first forest

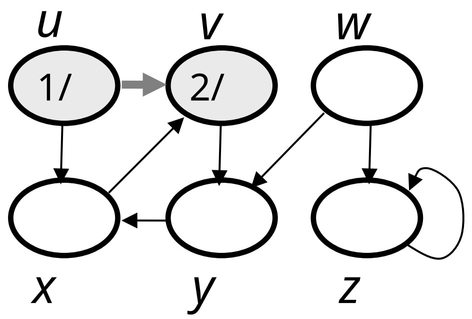
DFS-VISIT(u)

```
1.  color[u] ← GRAY
2.  time ← time+1
3.  d[u] ← time
4.  for each v ∈ Adj[u]
5.      do if color[v] = WHITE
6.            then π[v] ← u
7.                 DFS-VISIT(v)
8.  color[u] ← BLACK
9.  time ← time + 1
10. f[u] ← time
```

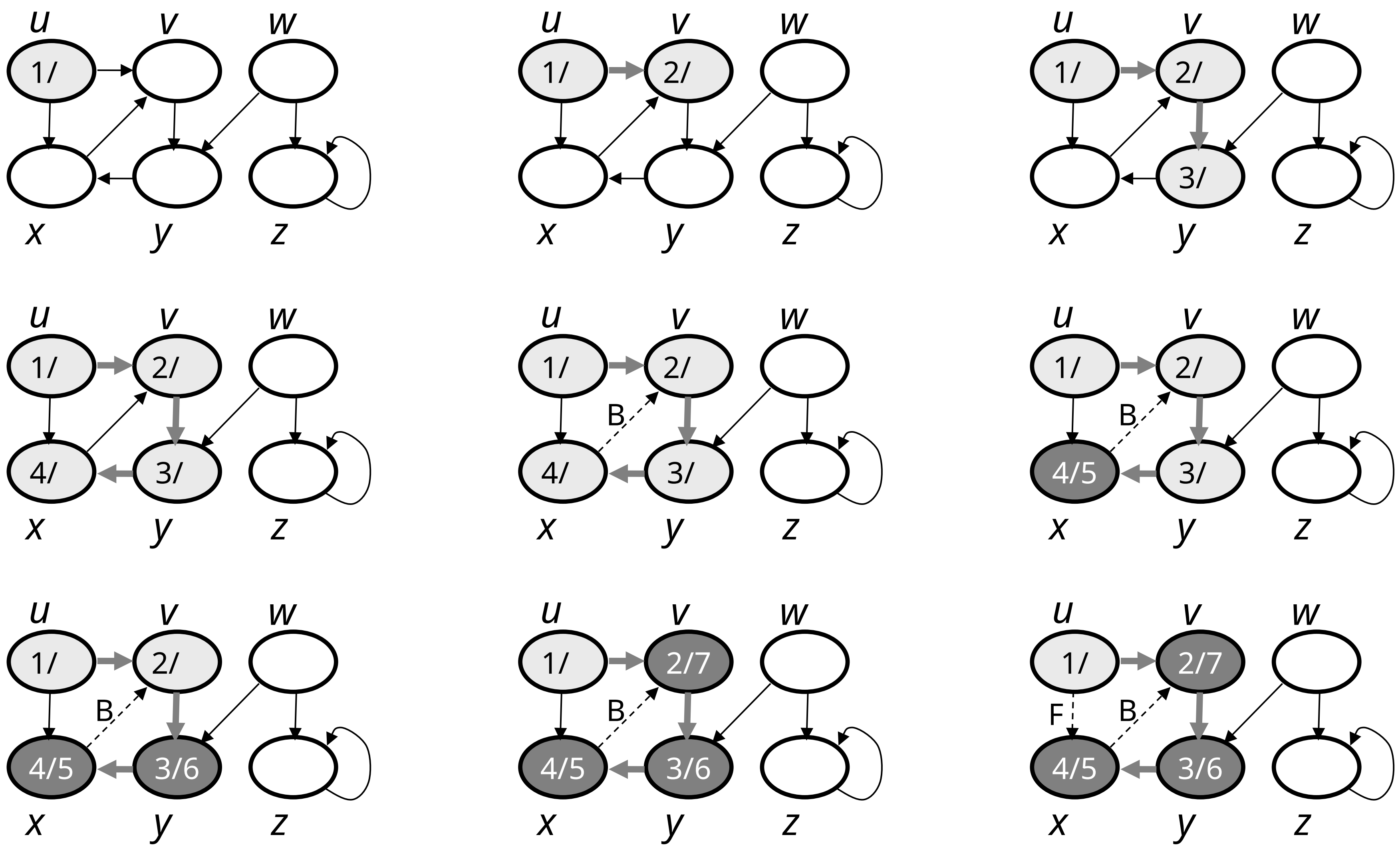


time = 1



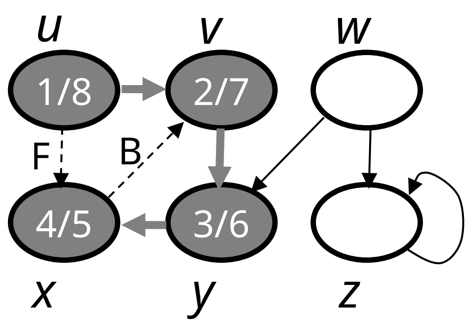


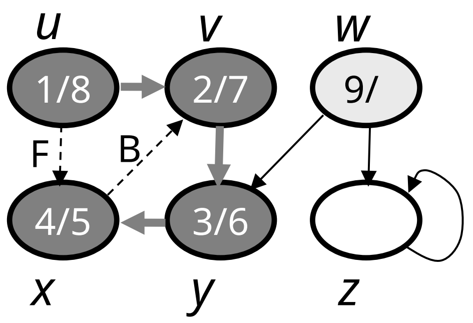
Example

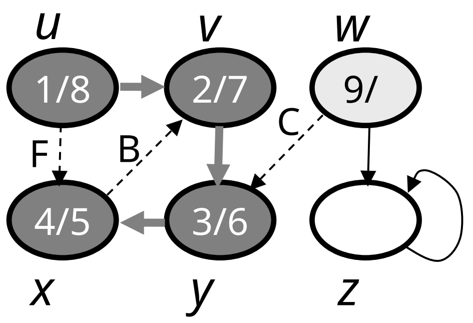


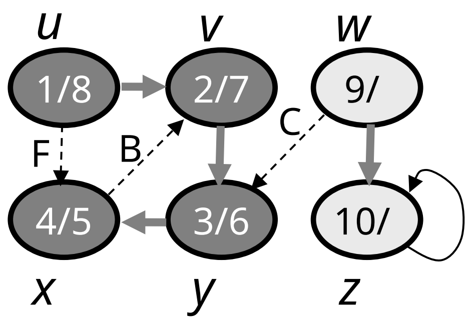
u v w
1/
x y z
u v w
1/ 2/
x y z
u v w
1/ 2/
3/
x y z
u v w
1/ 2/
4/ 3/
x y z
u v w
1/ 2/
B
4/ 3/
x y z
u v w
1/ 2/
B
4/5 3/
x y z
u v w
1/ 2/
B
4/5 3/6
x y z
u v w
1/ 2/7
B
4/5 3/6
x y z
u v w
1/ 2/7
F B
4/5 3/6
x y z

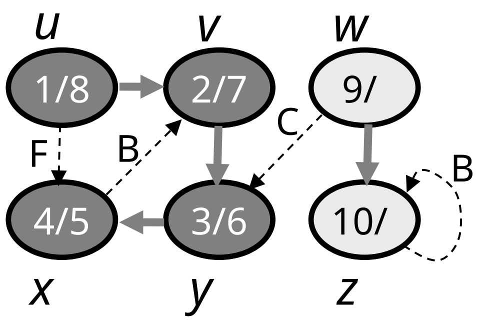
Example (cont.)

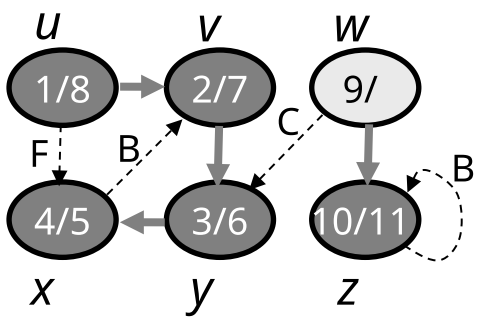


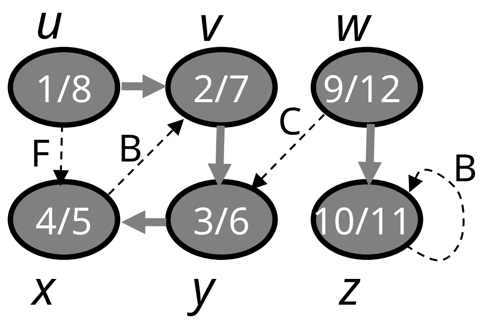










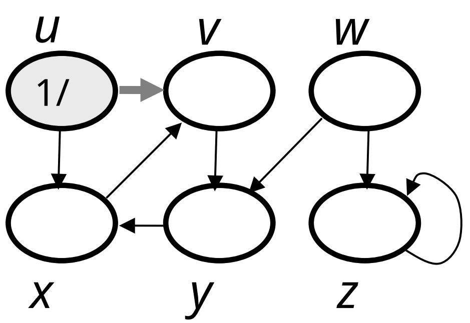


The results of DFS may depend on:

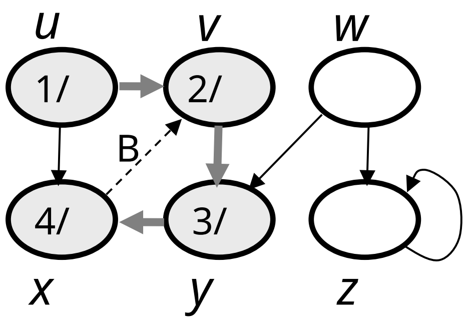
- The order in which nodes are explored in procedure DFS
- The order in which the neighbors of a vertex are visited in DFS-VISIT

Edge Classification

- **Tree edge** (reaches a WHITE vertex):
  - (u, v) is a tree edge if v was first discovered by exploring edge (u, v)

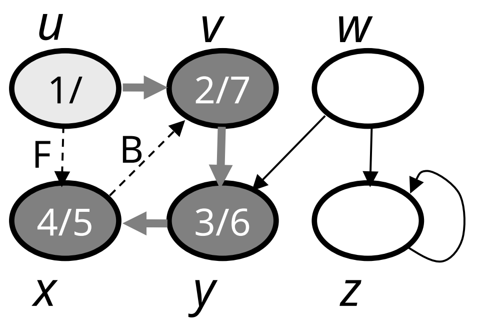


- **Back edge** (reaches a GRAY vertex):
  - (u, v), connecting a vertex u to an ancestor v in a depth first tree
  - Self loops (in directed graphs) are also back edges

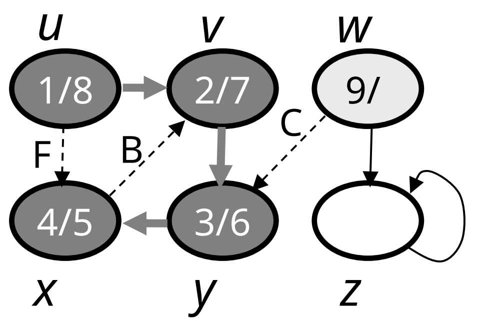


Edge Classification

- **Forward edge** (reaches a BLACK vertex & d[u] < d[v]):
  - Non-tree edge (u, v) that connects a vertex u to a descendant v in a depth first tree



- **Cross edge** (reaches a BLACK vertex & d[u] > d[v]):
  - Can go between vertices in same depth-first tree (as long as there is no ancestor / descendant relation) or between different depth-first trees



Analysis of DFS(V, E)

$$
\left.\begin{array}{l}
\textbf{1.}\quad \textbf{for} \text{ each } u \in V \\
\textbf{2.}\qquad\quad \textbf{do} \text{ color}[u] \leftarrow \text{WHITE} \\
3.\qquad\qquad\quad \pi[u] \leftarrow \text{NIL} \\
4.\quad \text{time} \leftarrow 0
\end{array}\right\} \Theta(|V|)
$$

$$
\left.\begin{array}{l}
\textbf{5.}\quad \textbf{for} \text{ each } u \in V \\
\textbf{6.}\qquad\quad \textbf{do if} \text{ color}[u] = \text{WHITE} \\
\textbf{7.}\qquad\qquad\qquad \textbf{then} \text{ DFS-VISIT}(u)
\end{array}\right\} \begin{array}{l}\Theta(|V|) \text{ – without} \\ \text{counting the time} \\ \text{for DFS-VISIT}\end{array}
$$

Analysis of DFS-VISIT(u)

```
1.  color[u] ← GRAY
2.  time ← time+1
3.  d[u] ← time
4.  for each v ∈ Adj[u]
5.      do if color[v] = WHITE
6.          then π[v] ← u
7.               DFS-VISIT(v)
8.  color[u] ← BLACK
9.  time ← time + 1
10. f[u] ← time
```

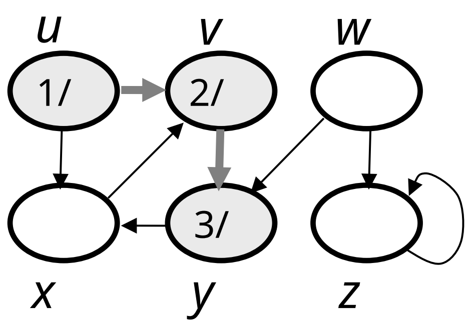
DFS-VISIT is called exactly once for each vertex

$\left.\right\}$ Each loop takes |Adj[u]|

$$\text{Total: } \underbrace{\Sigma_{u \in V} |Adj[u]|}_{\Theta(|E|)} + \Theta(|V|) = \Theta(|V| + |E|)$$

Properties of DFS

- u = π[v] ⟺ DFS-VISIT(v) was called during a search of u's adjacency list

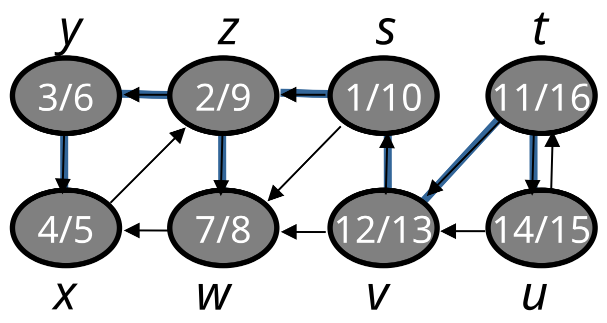


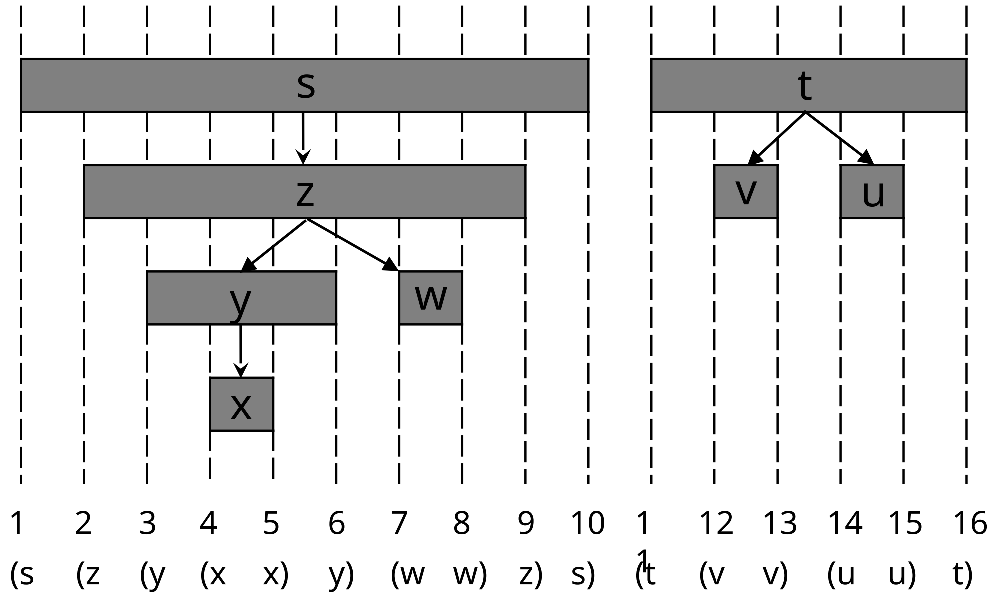
- Vertex v is a descendant of vertex u in the depth first forest ⟺ v is discovered during the time in which u is gray

Parenthesis Theorem

In any DFS of a graph G, for all u, v, exactly one of the following holds:

1. [d[u], f[u]] and [d[v], f[v]] are disjoint, and neither of u and v is a descendant of the other
2. [d[v], f[v]] is entirely within [d[u], f[u]] and v is a descendant of u
3. [d[u], f[u]] is entirely within [d[v], f[v]] and u is a descendant of v





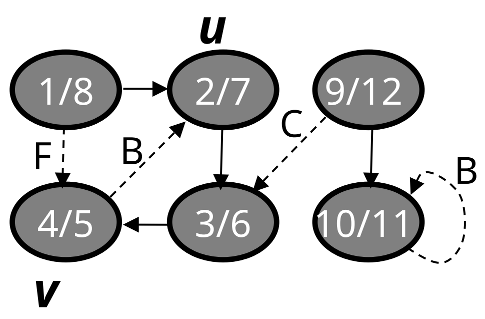
Well-formed expression: parenthesis are properly nested

Other Properties of DFS

Corollary

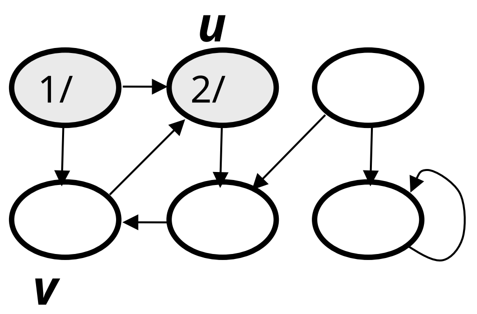
Vertex v is a proper descendant of u

$\Longleftrightarrow$ d[u] < d[v] < f[v] < f[u]



Theorem (White-path Theorem)

In a depth-first forest of a graph G, vertex v is a descendant of u if and only if at time d[u], there is a path u v consisting of only white vertices.



Topological Sort

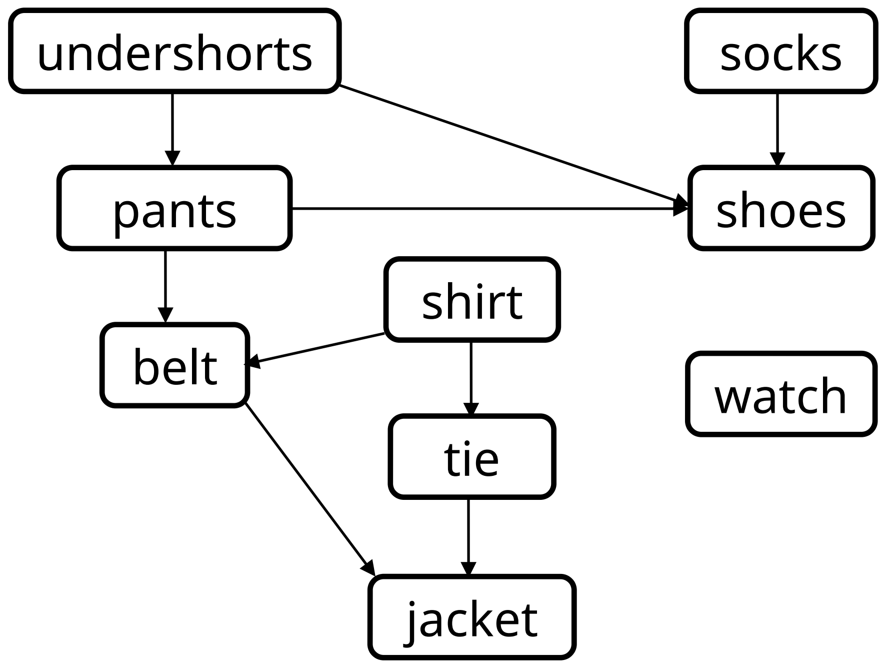
Topological sort of a directed acyclic graph G = (V, E): a linear order of vertices such that if there exists an edge (u, v), then u appears before v in the ordering.

- Directed acyclic graphs (DAGs)
  - Used to represent precedence of events or processes that have a **partial order**

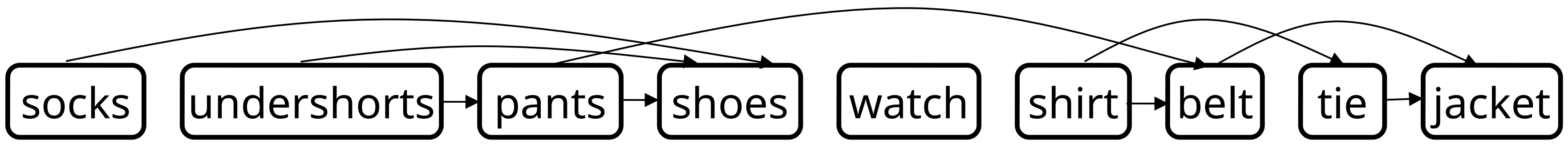
$$\left.\begin{array}{l}\textbf{a}\text{ before }\textbf{b}\\ \textbf{b}\text{ before }\textbf{c}\end{array}\right\}\ \textbf{a}\text{ before }\textbf{c} \qquad \left.\begin{array}{l}\textbf{b}\text{ before }\textbf{c}\\ \textbf{a}\text{ before }\textbf{c}\end{array}\right\}\ \begin{array}{l}\text{What about}\\ \textbf{a}\text{ and }\textbf{b}\text{?}\end{array}$$

Topological sort helps us establish a **total order**

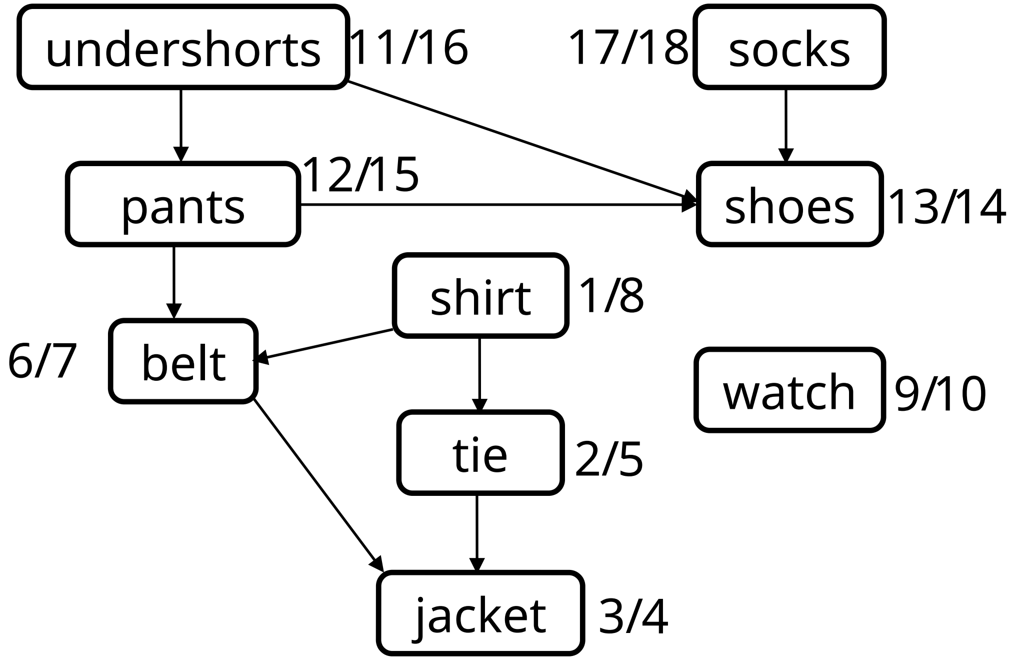
Topological Sort



Topological sort:
an ordering of vertices along a horizontal line so that all directed edges go from left to right.

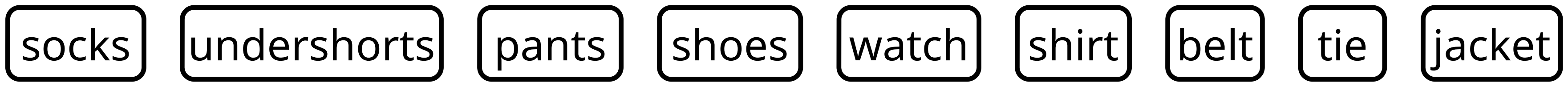


Topological Sort



TOPOLOGICAL-SORT(V, E)

1. Call DFS(V, E) to compute finishing times f[v] for each vertex v
2. When each vertex is finished, insert it onto the front of a linked list
3. Return the linked list of vertices



Running time: Θ(|V| + |E|)

Lemma

A directed graph is **acyclic** ⟺ a DFS on G yields no back edges.

Proof:

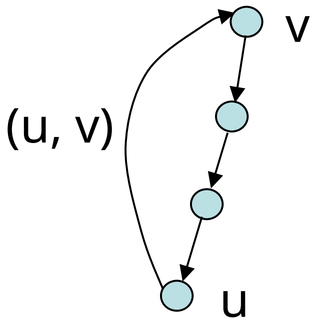
“⇒”: acyclic ⇒ no back edge

- Assume **back edge** ⇒ prove **cycle**
- Assume there is a back edge (u, v)

⇒ v is an ancestor of u

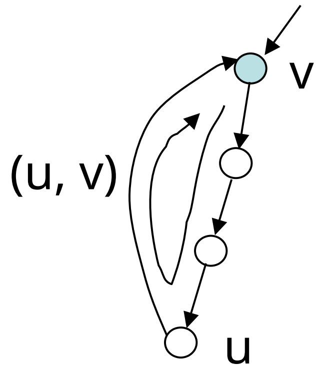
⇒ there is a path from v to u in G (v u)

⇒ v u + the back edge (u, v) yield a cycle



Lemma

A directed graph is **acyclic** ⟺ a DFS on G yields no back edges.



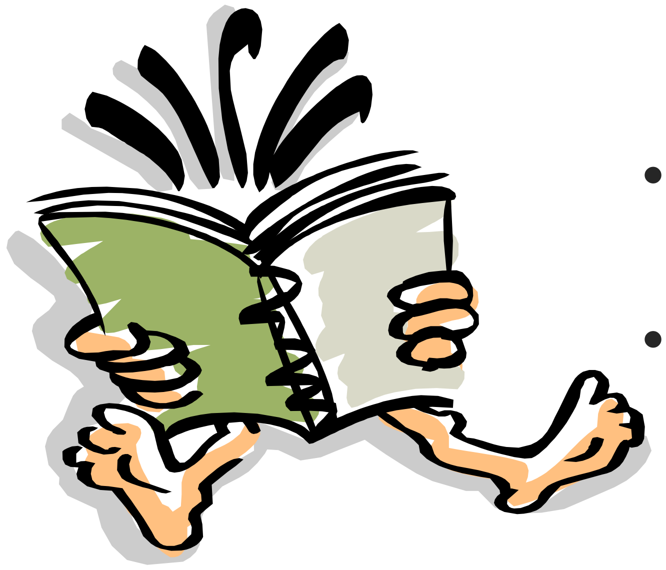
Proof:

"⟸": no back edge ⇒ acyclic

- Assume **cycle** ⇒ prove **back edge**
- Suppose G contains cycle c
- Let v be the first vertex discovered in c, and (u, v) be the preceding edge in c
- At time d[v], vertices of c form a white path v u
- u is descendant of v in depth-first forest (by white-path theorem)

⇒ (u, v) is a back edge

Readings



- For this lecture
  - Chapter 15
- Coming next
  - Chapter 20