Analysis of Algorithms
CS 477/677

Instructor: Monica Nicolescu

Lecture 26

NP-Completeness

- **Polynomial-time algorithms**
  on inputs of size n, worst-case running time is $O(n^k)$, for a constant k
- Not all problems can be solved in polynomial time
  - Some problems cannot be solved by any computer no matter how much time is provided (Turing's Halting problem) – such problems are called **undecidable**
  - Some problems can be solved but not in $O(n^k)$

Class of "P" Problems

- **Class P** consists of (decision) problems that are solvable in polynomial time:

  there exists an algorithm that can solve the problem in $O(n^k)$, k constant

- Problems in P are also called **tractable**
- Problems not in P are also called **intractable**
  - Can be solved in reasonable time only for small inputs

Optimization & Decision Problems

- **Decision problems**
  - Given an input and a question regarding a problem, determine if the answer is yes or no
- **Optimization problems**
  - Find a solution with the "best" value
- Optimization problems can be cast as decision problems that are easier to study
  - E.g.: Shortest path: G = unweighted directed graph
    - Find a path between u and v that uses the fewest edges
    - *Does a path exist from u to v consisting of at most k edges?*

Nondeterministic Algorithms

Nondeterministic algorithm = two stage procedure:

1) Nondeterministic (“guessing”) stage:

   generate an arbitrary string that can be thought of as a candidate solution (“certificate”)

2) Deterministic (“verification”) stage:

   take the certificate and the instance to the problem and return YES if the certificate represents a solution

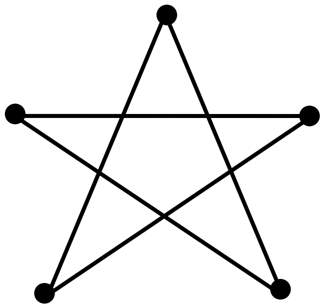
- **Nondeterministic polynomial** (NP) = verification stage is polynomial

Class of “NP” Problems

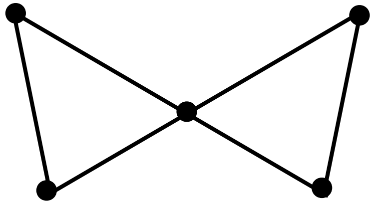
- **Class NP** consists of problems that are verifiable in polynomial time (i.e., could be solved by nondeterministic polynomial algorithms)
  - If we were given a “certificate” of a solution, we could verify that the certificate is correct in time polynomial to the size of the input

E.g.: Hamiltonian Cycle

- **Given:** a directed graph G = (V, E), determine a simple cycle that contains each vertex in V
  - Each vertex can only be visited once
- **Certificate**:
  - Sequence: $\langle v_1, v_2, v_3, \ldots, v_{|V|} \rangle$
- **Verification**:
  1) $(v_i, v_{i+1}) \in E$ for $i = 1, \ldots, |V|$
  2) $(v_{|V|}, v_1) \in E$

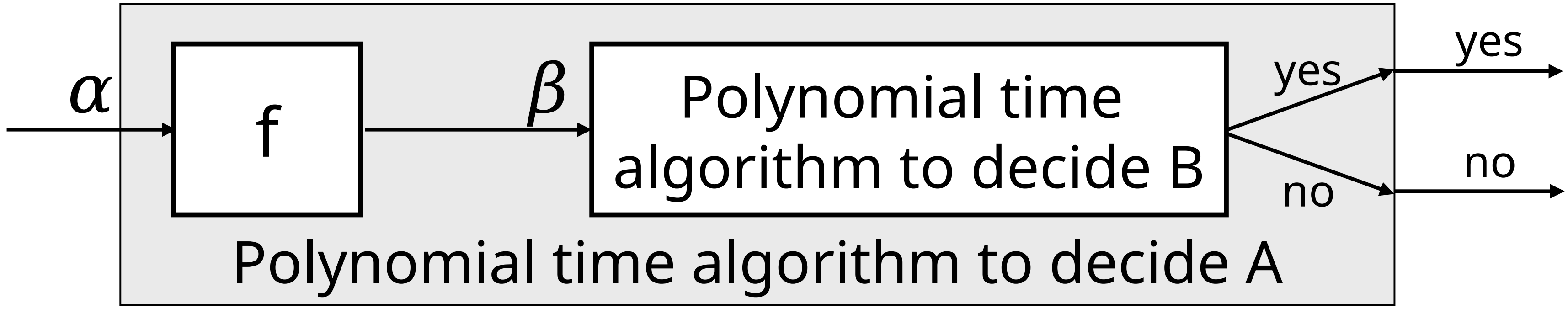


hamiltonian



not hamiltonian

Polynomial Reduction Algorithm



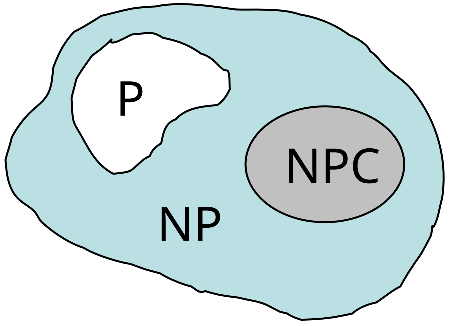
- To solve a decision problem A in polynomial time
  1. Use a polynomial time reduction algorithm to transform A into B
  2. Run a known polynomial time algorithm for B
  3. Use the answer for B as the answer for A

Reductions

- Given two problems A, B, we say that A is **reducible** to B ($A \leq_p B$) if:

1. There exists a function f that converts the input of A to an input of B in polynomial time
2. A(i) = YES $\Longleftrightarrow$ B(f(i)) = YES (for every input i)

NP-Completeness



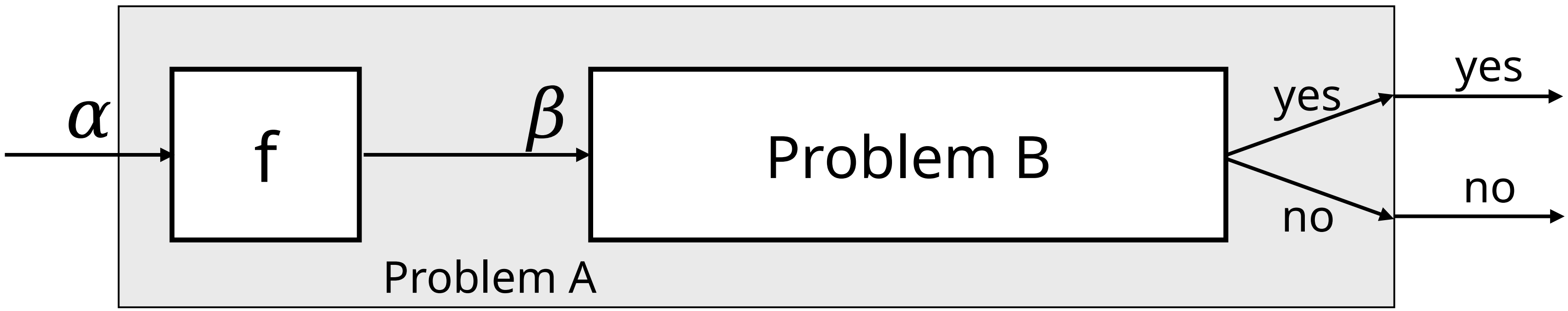
- A problem B is **NP-complete (NPC)** if:

  1) $B \in$ **NP**

  2) $A \leq_p B$ for all $A \in$ **NP**

- If B satisfies only property 2) we say that B is **NP-hard**
- No polynomial time algorithm has been discovered for an **NP-Complete** problem
- No one has ever proven that no polynomial time algorithm can exist for any **NP-Complete** problem

Reduction and NP-Completeness



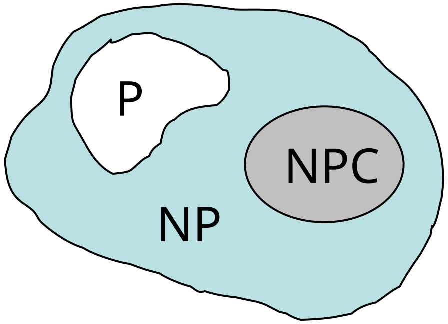
- Suppose we know:
  - No polynomial time algorithm exists for problem A
  - We have a polynomial reduction f from A to B

$\Rightarrow$ No polynomial time algorithm exists for B

Proving NP-Completeness

Theorem: If A is NP-Complete and $A \leq_p B$

$\Rightarrow$ B is NP-Hard



In addition, if $B \in NP$

$\Rightarrow$ B is NP-Complete

Proof: Assume that $B \in P$

Since $A \leq_p B \Rightarrow A \in P$ contradiction, so $B \notin P$

If B $\in NP \Rightarrow B \in$ NP-Complete (by definition of NP-C)

If B $\notin NP \Rightarrow B \in$ NP-Hard (by definition of NP-H)

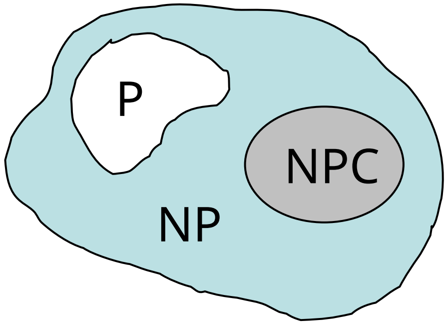
Proving NP-Completeness

1. Prove that the problem B is in NP
   – A randomly generated string can be checked in polynomial time to determine if it represents a solution
2. Show that **one known** NP-Complete problem can be transformed to B in polynomial time
   – No need to check that **all** NP-Complete problems are reducible to B

Is P = NP?

- Any problem in P is also in NP:

$$P \subseteq NP$$



- We can solve problems in P, even without having a certificate
- The big (and open question) is whether P = NP

Theorem: If any NP-Complete problem can be solved in polynomial time ⇒ then P = NP.

P & NP-Complete Problems

- **Shortest simple path**
  - Given a graph G = (V, E) find a **shortest** path from a source to all other vertices
  - Polynomial solution: O(VE)
- **Longest simple path**
  - Given a graph G = (V, E) find a **longest** path from a source to all other vertices
  - NP-complete

P & NP-Complete Problems

- **Euler tour**
  - Given G = (V, E) a connected, directed graph, find a cycle that traverses each edge of G exactly once (may visit a vertex multiple times)
  - Polynomial solution O(E)
- **Hamiltonian cycle**
  - G = (V, E) a connected, directed graph find a cycle that visits each vertex of G exactly once
  - NP-complete

Boolean Formula Satisfiability

Formula Satisfiability Problem: a boolean formula Φ composed of

1. n boolean variables: $x_1, x_2, \ldots, x_n$
2. m boolean connectives: $\wedge$ (AND), $\vee$ (OR), $\neg$ (NOT), $\rightarrow$ (implication), $\longleftrightarrow$ (equivalence, "if and only if")
3. Parentheses

Satisfying assignment: an assignment of values (0, 1) to variables x_i that causes Φ to evaluate to 1

E.g.: $\Phi = (x_1 \vee x_2) \wedge (x_1 \vee \neg x_2) \wedge (\neg x_1 \vee \neg x_2)$

Certificate: $x_1 = 1, x_2 = 0 \Rightarrow \Phi = 1 \wedge 1 \wedge 1 = 1$

– Formula Satisfiability is first to be proven NP-Complete

3-CNF Satisfiability

3-CNF (clause normal form) Satisfiability Problem:

- n boolean variables: $x_1, x_2, \ldots, x_n$
- **Literal**: x_i or $\neg x_i$ (a variable or its negation)
- **Clause:** c_j = an **OR** of **three literals**
- Formula: $\boldsymbol{\Phi} = c_1 \wedge c_2 \wedge \ldots \wedge c_m$ (m clauses)

- E.g.:

$$\boldsymbol{\Phi} = (x_1 \vee \neg x_1 \vee \neg x_2) \wedge (x_3 \vee x_2 \vee x_4) \wedge (\neg x_1 \vee \neg x_3 \vee \neg x_4)$$

- **3-CNF** is NP-Complete

Clique

Clique Problem:

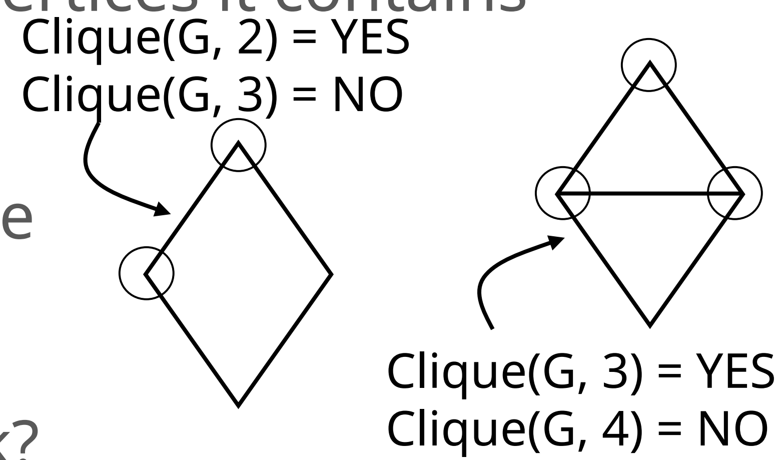
- Undirected graph G = (V, E)
- **Clique:** a subset of vertices in V all connected to each other by edges in E (i.e., forming a complete graph)
- **Size of a clique:** number of vertices it contains

Optimization problem:

- Find a clique of maximum size

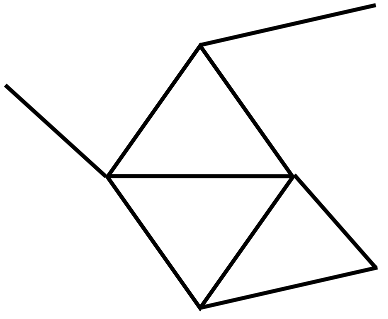
Decision problem:

- Does G have a clique of size k?



Clique Verifier

- **Given**: an undirected graph G = (V, E)
- **Problem**: Does G have a clique of size k?
- **Certificate**:
  - A set of k nodes



- **Verifier**:
  - Verify that for all pairs of vertices in this set there exists an edge in E
- Let's prove that the clique problem is NP-Complete

3-CNF $\leq_p$ Clique

- Start with an instance of 3-CNF:
  - $\Phi = C_1 \land C_2 \land \ldots \land C_k$ (k clauses)
  - Each clause C_r has three literals: $C_r = l_1^r \lor l_2^r \lor l_3^r$
- **Idea:**
  - Construct a graph G such that Φ is satisfiable if and only if G has a clique of size k

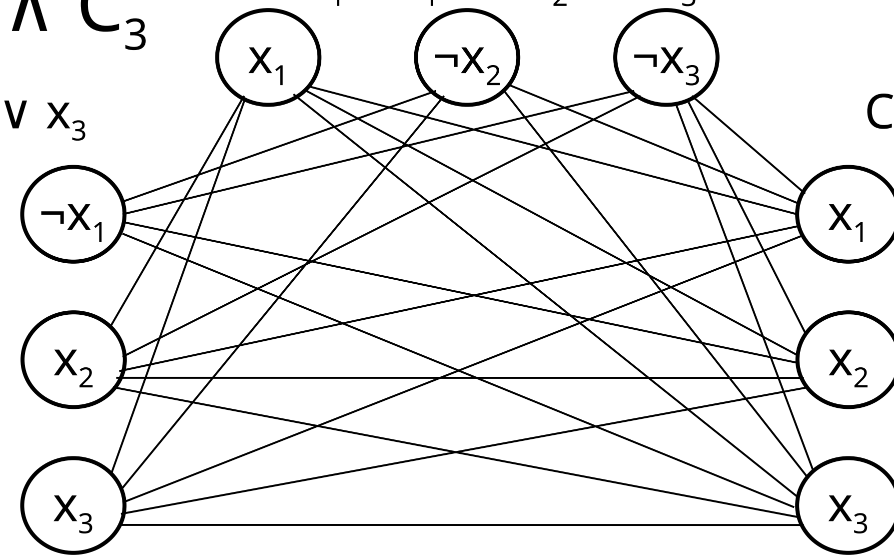
3-CNF $\leq_p$ Clique

$\Phi = C_1 \wedge C_2 \wedge C_3$

$C_1 = x_1 \vee \neg x_2 \vee \neg x_3$

$C_2 = \neg x_1 \vee x_2 \vee x_3$

$C_3 = x_1 \vee x_2 \vee x_3$

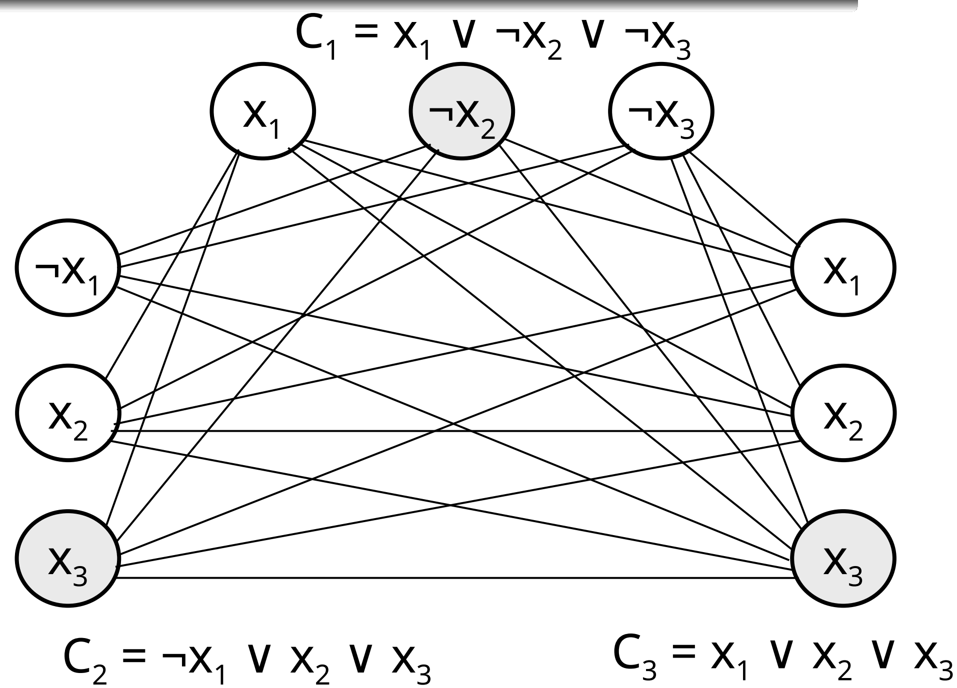


- For each clause $C_r = l_1^r \vee l_2^r \vee l_3^r$ place a triple of vertices v_1^r, v_2^r, v_3^r in V
- Put an edge between two vertices v_i^r and v_j^s if:
  - v_i^r and v_j^s are in different triples
  - l_i^r is not the negation of l_j^s

3-CNF $\leq_p$ Clique

$\Phi = C_1 \wedge C_2 \wedge C_3$

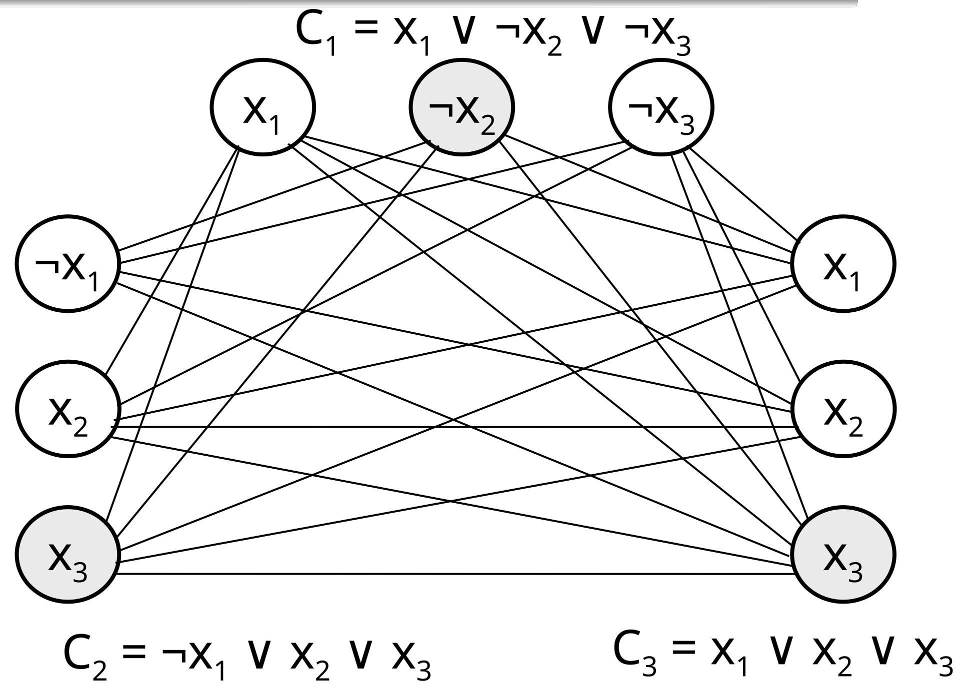
- Suppose Φ has a satisfying assignment
  - Each clause C_r has some literal assigned to 1 – this corresponds to a vertex v_i^r
  - Picking one such literal from each C_r ⇒ a set V' of k vertices



- Claim: V' is a clique
  - $\forall\, v_i^r, v_j^s \in V'$ the corresponding literals are 1 ⇒ cannot be complements
  - by the design of G the edge $(v_i^r, v_j^s) \in E$

3-CNF $\leq_p$ Clique

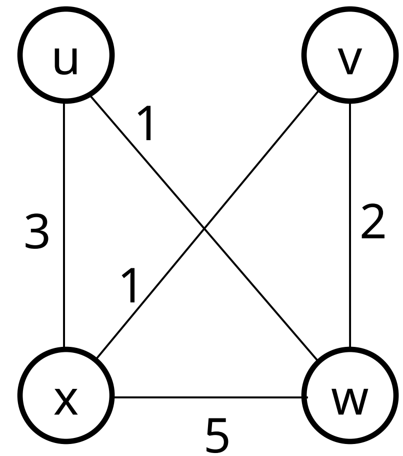
$\boldsymbol{\Phi} = C_1 \wedge C_2 \wedge C_3$



- Suppose *G* has a clique of size k
  - No edges between nodes in the same clause
  - Clique contains only one vertex from each clause
  - Assign 1 to vertices in the clique (we can do it because the literals of these vertices cannot belong to complementary literals)
  - Each clause is satisfied ⇒ $\boldsymbol{\Phi}$ is satisfied

The Traveling Salesman Problem

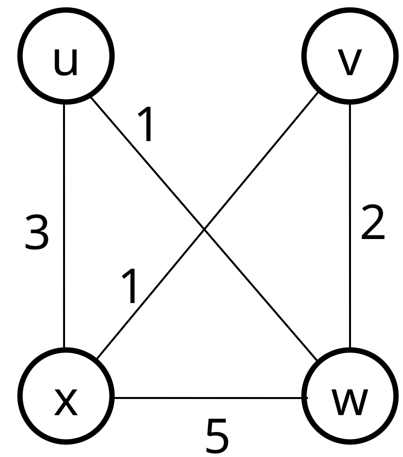
- G = (V, E), |V| = n, vertices represent cities
- **Cost**: c(i, j) = cost of travel from city i to city j
- **Problem**: salesman should make a tour (hamiltonian cycle):
  - Visit each city only once
  - Finish at the city he started from
  - Total cost is minimum
- TSP = tour with cost at most k



⟨u, w, v, x⟩

TSP $\in$ NP

- **Certificate:**
  - Sequence of n vertices, cost
  - E.g.: $\langle u, w, v, x \rangle$, 7
- **Verification:**
  - Each vertex occurs only once
  - Sum of costs is at most k

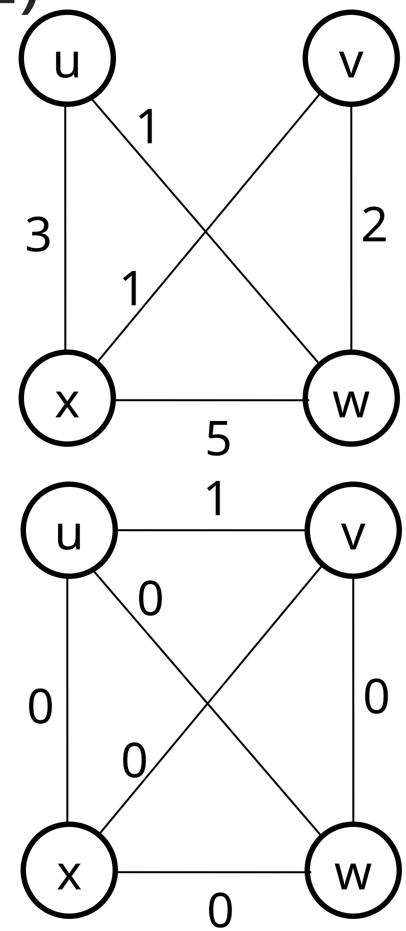


HAM-CYCLE $\leq_p$ TSP

- Start with a Hamiltonian cycle G = (V, E)
- Form the complete graph G' = (V, E')

$$E' = \{(i, j): i, j \in V \text{ and } i \neq j\}$$

$$c(i, j) = \begin{cases} 0 & \text{if } (i, j) \in E \\ 1 & \text{if } (i, j) \notin E \end{cases}$$



- Let's prove that:
- G has a hamiltonian cycle $\Longleftrightarrow$

  G' has a tour of cost at most 0

HAM-CYCLE $\leq_p$ TSP

- G has a hamiltonian cycle h
  - ⇒ Each edge in h ∈ E ⇒ has cost 0 in G'
  - ⇒ h is a tour in G' with cost 0
- G' has a tour h' of cost at most 0
  - ⇒ Each edge on tour must have cost 0
  - ⇒ h' contains only edges in E

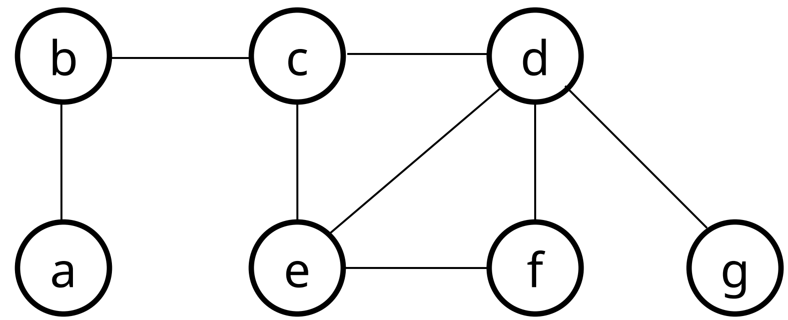
Approximation Algorithms

Various ways to get around NP-completeness:

1. If inputs are small, an algorithm with exponential time may be satisfactory
2. Isolate special cases, solvable in polynomial time
3. Find near-optimal solutions in polynomial time
   - Approximation algorithms
   - Local search (hill climbing)

The Vertex-Cover Problem

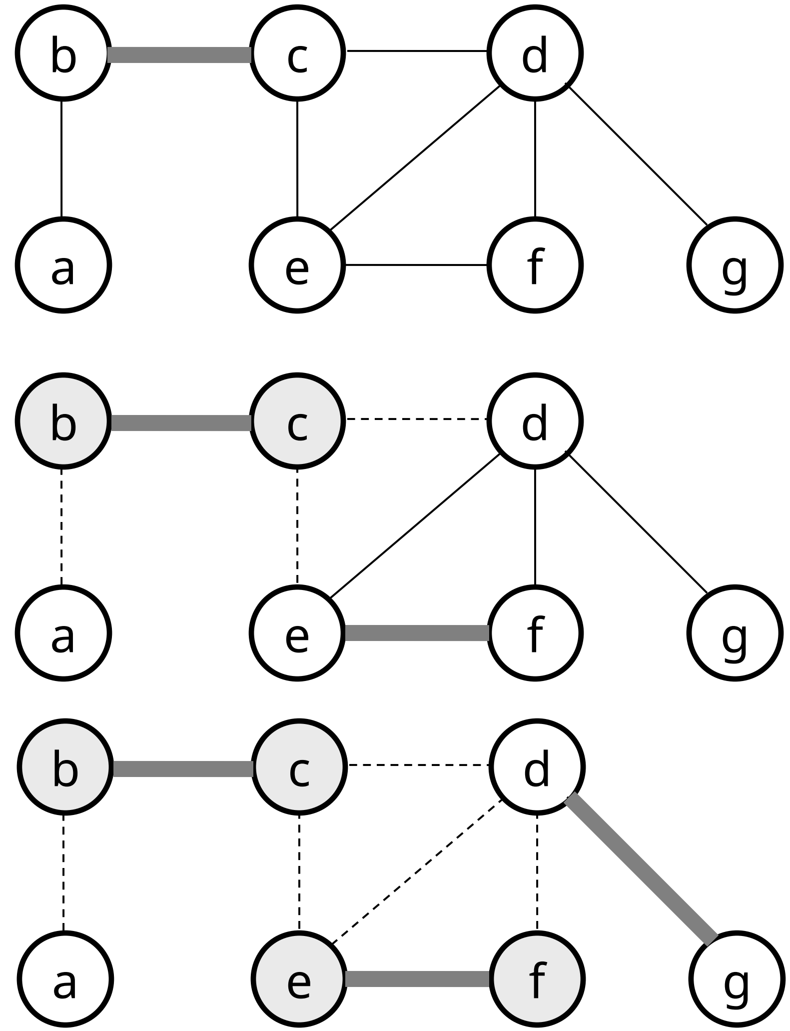
- Vertex cover of G = (V, E), undirected graph
  - A subset V’ ⊆ V that covers all the edges in G



- **Approximate solution (greedy)**:
  - Start with a list of all edges
  - Repeatedly pick an arbitrary edge (u, v)
  - Add its endpoints u and v to the vertex-cover set
  - Remove from the list all edges incident on u or v

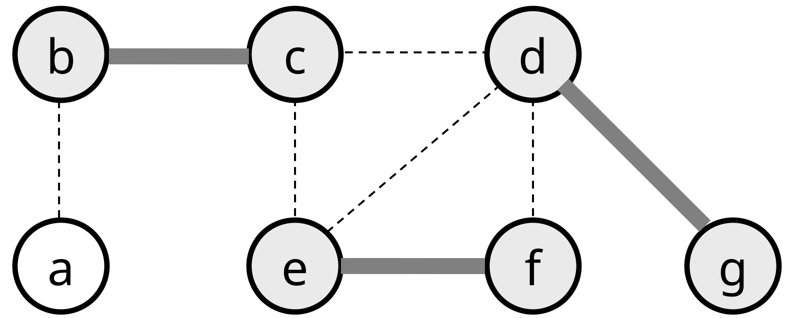
APPROX-VERTEX-COVER(G)

1. $C \leftarrow \emptyset$
2. $E' \leftarrow E[G]$
3. **while** $E' \neq \emptyset$
4. **do** choose (u, v) arbitrary from E'
5. $C \leftarrow C \cup \{u, v\}$
6. remove from E' all edges incident on u, v
7. **return** C

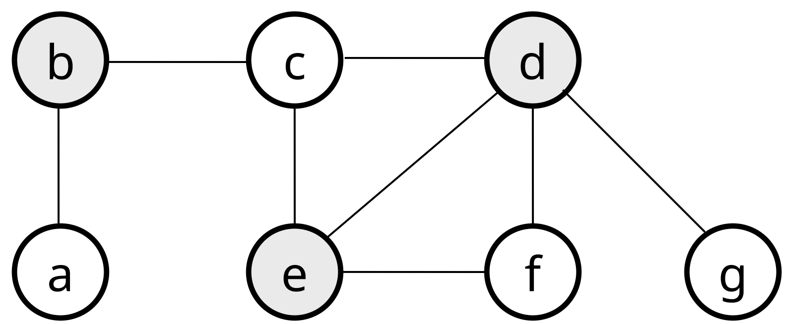


APPROX-VERTEX-COVER(G)

APPROX-VERTEX-COVER:



Optimal VERTEX-COVER:



It can be proven that the approximation algorithm returns a solution that is no more than twice the optimal vertex cover.

The Set Covering Problem

- Finite set X
- Family F of subsets of X: $F = \{S_1, S_2, \ldots, S_n\}$

$$X = \bigcup_{S \in F} S$$

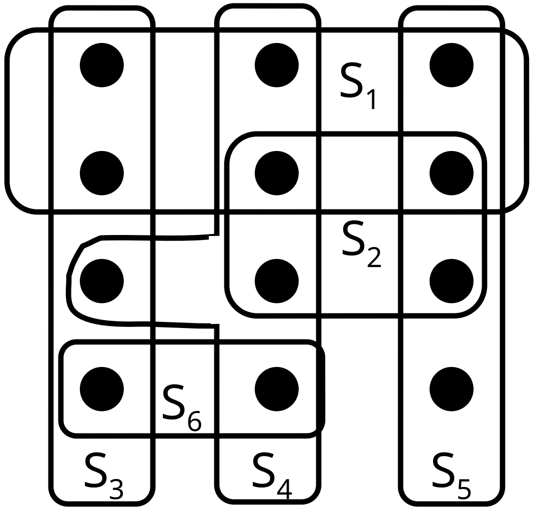
- Find a minimum-size subset $C \subseteq F$ that covers all the elements in X
- Decision: given a number k find if there exist k sets $S_{i1}, S_{i2}, \ldots, S_{ik}$ such that:

$$S_{i1} \cup S_{i2} \cup \ldots \cup S_{ik} = X$$

Greedy Set Covering

Idea:

- At each step pick a set S that covers the greatest number of remaining elements



Optimal: C = {S_3, S_4, S_5}

GREEDY-SET-COVER(X, F)

$U \leftarrow X$

$C \leftarrow \emptyset$

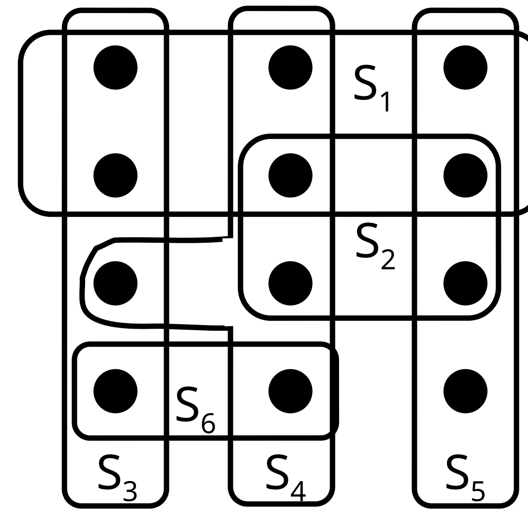
while $U \neq \emptyset$

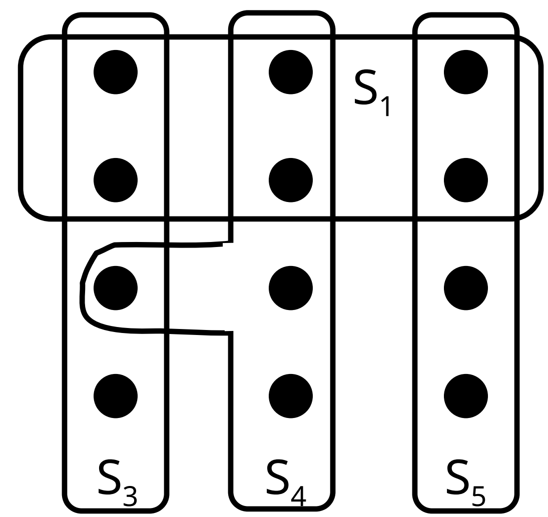
do select an $S \in F$ that maximizes $|S \cap U|$

$U \leftarrow U - S$

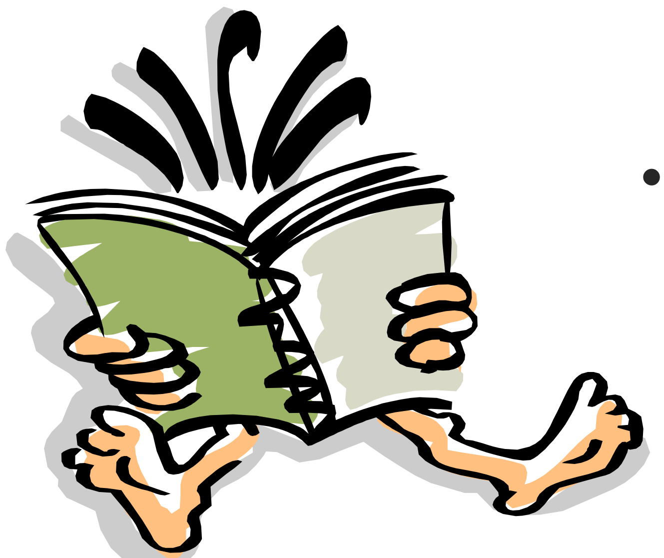
$C \leftarrow C \cup \{S\}$

return C





Readings

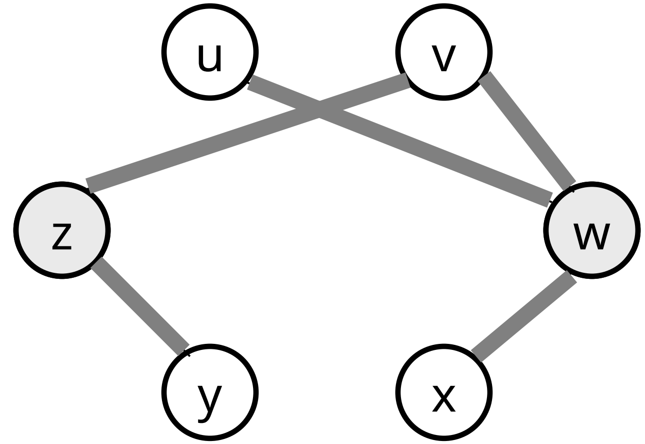


- Chapters 25, 31

Optional, not required for final exam

ADDITIONAL PROOFS

Vertex Cover

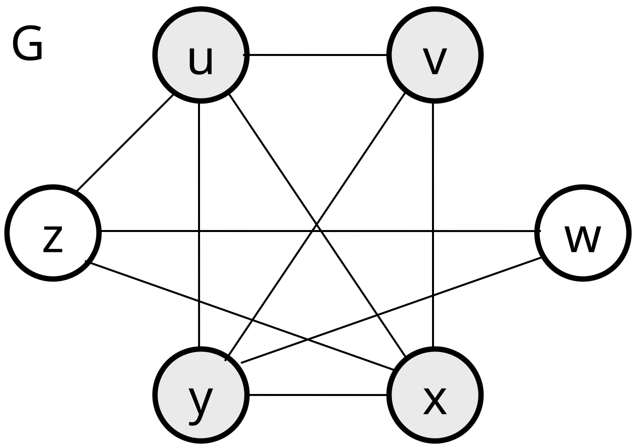


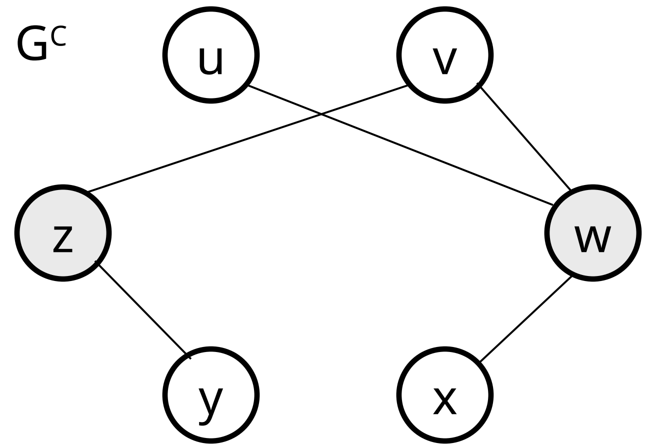
- G = (V, E), undirected graph
- **Vertex cover** = a subset V’ ⊆ V which covers all the edges
  - if (u, v) ∈ E then u ∈ V’ or v ∈ V’ or both.
- **Size** of a vertex cover = number of vertices in it

Problem:

- Find a vertex cover of minimum size
- Does graph G have a vertex cover of size k?

Clique $\leq_p$ Vertex Cover





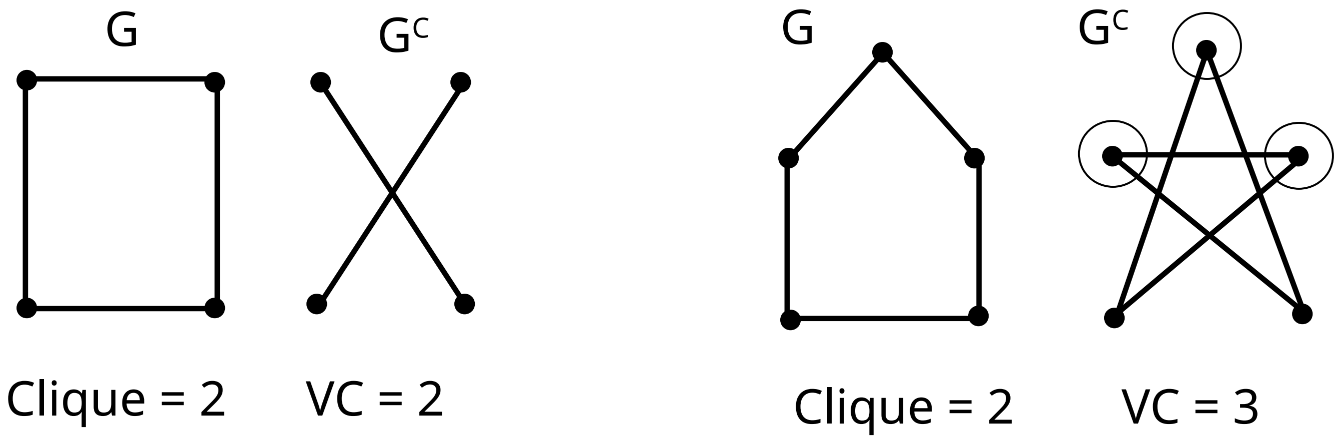
- $G = (V, E) \Rightarrow$ complement graph $G^C = (V, E^C)$

  $E^C = \{(u, v):, u, v \in V, \text{ and } (u, v) \notin E\}$

Idea:

$\langle G, k\rangle$ (clique) $\rightarrow \langle G^C, |V|\text{-}k\rangle$ (vertex cover)

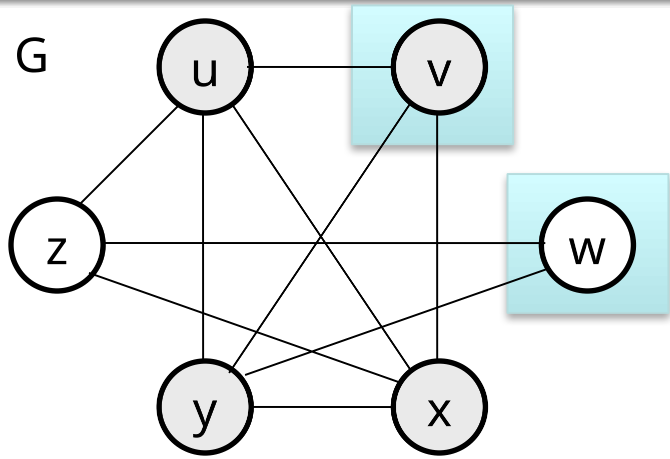
Clique $\leq_p$ Vertex Cover (VC)

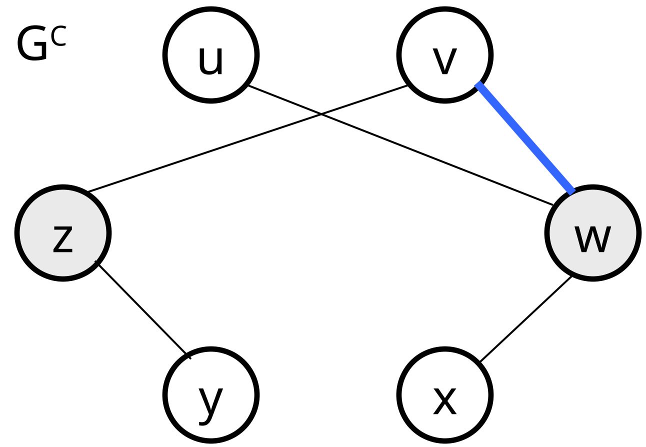


Size[Clique](G) + Size[Vertex Cover](G^C) = n

- G has a **clique** of size k $\Longleftrightarrow$ G^C has a **vertex cover** of size n – k
- S is a clique in G $\Longleftrightarrow$ V – S is a vertex cover in G^C

Clique $\leq_p$ Vertex Cover





- Prove: G has a clique $V' \subseteq V$, $|V'| = k \Rightarrow V\text{-}V'$ is a VC in G^C
- Let $(v, w) \in E^C \quad \Rightarrow (v, w) \notin E$

$\Rightarrow$ v and w were not connected in E

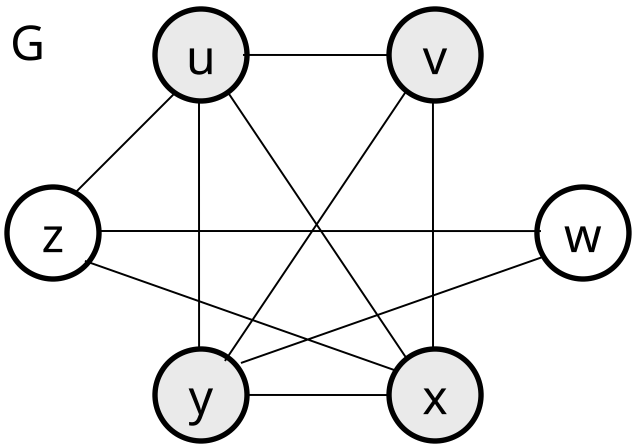
$\Rightarrow$ at least one of v or w does not belong in the clique V'

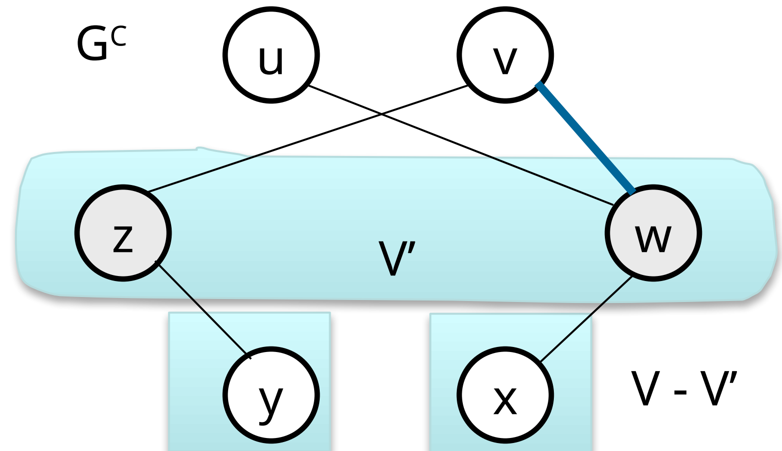
$\Rightarrow$ at least one of v or w belongs in V - V'

$\Rightarrow$ edge (v, w) is covered by V – V'

$\Rightarrow$ edge (v, w) was arbitrary $\Rightarrow$ every edge of E^C is covered

Clique $\leq_p$ Vertex Cover





- Prove: G^C has a vertex cover $V' \subseteq V$, $|V'| = |V| - k \Rightarrow V\text{-}V'$ is a clique in G
- For all v, w $\in$ V, if (v, w) $\in E^C$

$\Rightarrow$ v $\in$ V' or w $\in$ V' or both $\in$ V'

$\Rightarrow$ For all x, y $\in$ V, if x $\notin$ V' and y $\notin$ V':

$\Rightarrow$ no edge between x, y in E^G $\Rightarrow$ (x,y) $\in$ E

$\Rightarrow$ V – V' is a clique, of size $|V| - |V'| = k$

The Traveling Salesman Problem

- G = (V, E), |V| = n, vertices represent cities
- **Cost**: c(i, j) = cost of travel from city i to city j
- **Problem**: salesman should make a tour (hamiltonian cycle):
  - Visit each city only once
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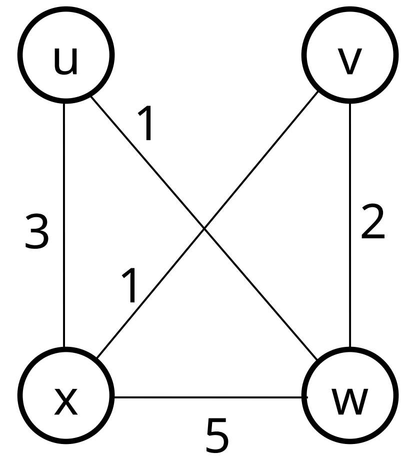
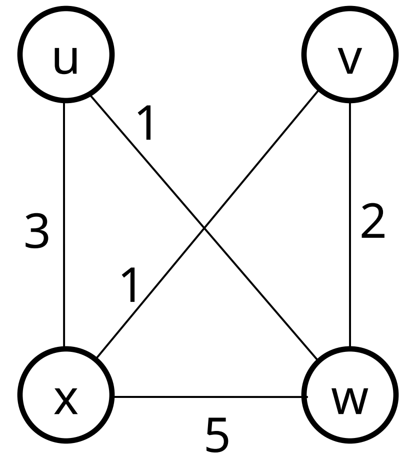
  - Total cost is minimum
- TSP = tour with cost at most k



⟨u, w, v, x⟩

TSP ∈ NP

- **Certificate:**
  - Sequence of n vertices, cost
  - E.g.: ⟨u, w, v, x⟩, 7
- **Verification:**
  - Each vertex occurs only once
  - Sum of costs is at most k

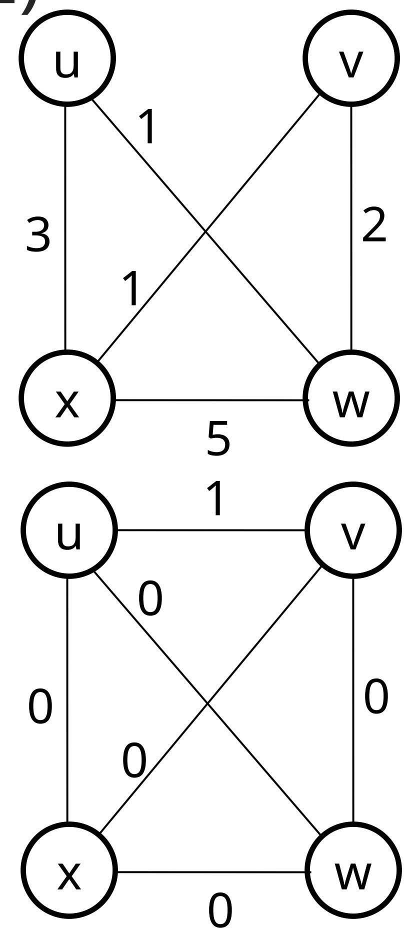


HAM-CYCLE $\leq_p$ TSP

- Start with a Hamiltonian cycle G = (V, E)
- Form the complete graph G' = (V, E')

$$E' = \{(i, j): i, j \in V \text{ and } i \neq j\}$$

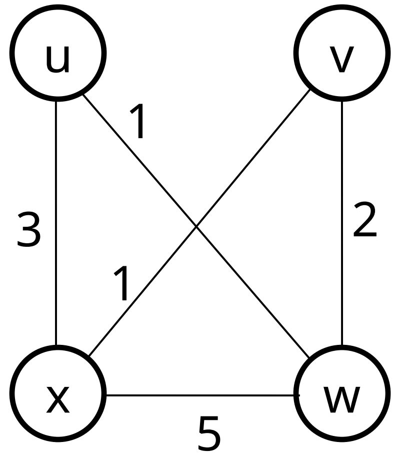
$$c(i, j) = \begin{cases} 0 & \text{if } (i, j) \in E \\ 1 & \text{if } (i, j) \notin E \end{cases}$$

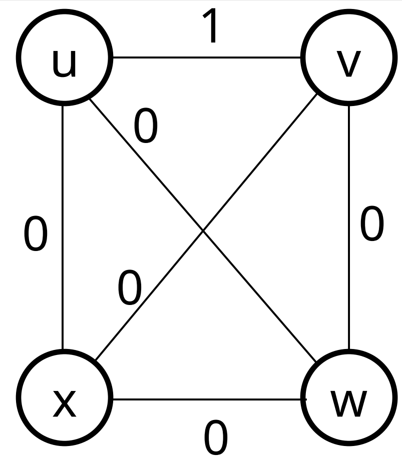


- Let's prove that:
- G has a hamiltonian cycle $\Longleftrightarrow$

  G' has a tour of cost at most 0

HAM-CYCLE $\leq_p$ TSP



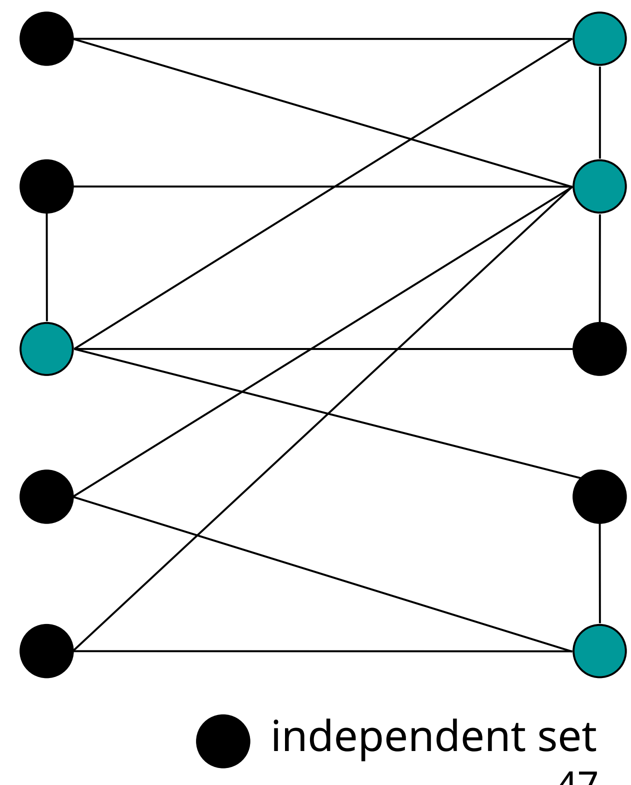


- G has a hamiltonian cycle h
  - ⇒ Each edge in h ∈ E ⇒ has cost 0 in G'
  - ⇒ h is a tour in G' with cost 0
- G' has a tour h' of cost at most 0
  - ⇒ Each edge on tour must have cost 0
  - ⇒ h' contains only edges in E

INDEPENDENT-SET

- Given a graph G = (V, E) and an integer k, is there a subset of vertices $S \subseteq V$ such that $|S| \geq k$, and for each edge at most one of its endpoints is in S?

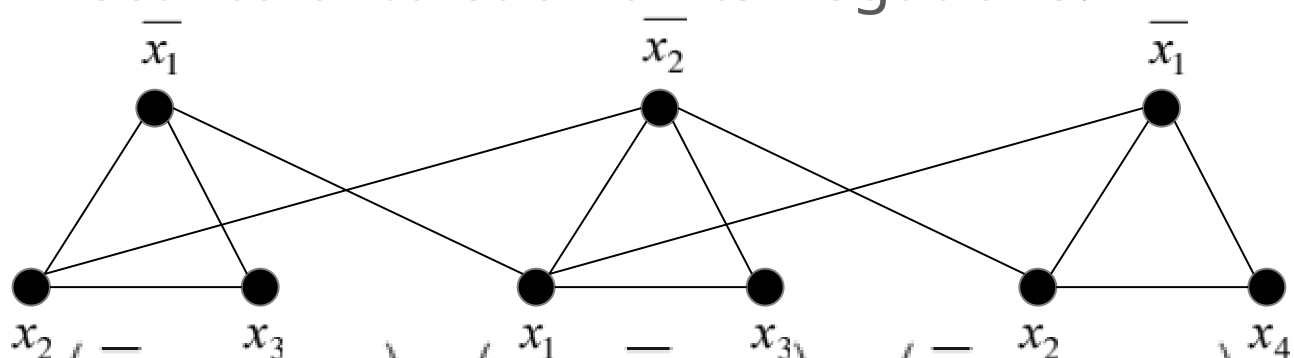
- Is there an independent set of size ≥ 6?
  - Yes.
- Is there an independent set of size ≥ 7?
  - No.



3-CNF $\leq_p$ INDEPENDENT-SET

- Given an instance Φ of 3-CNF, we construct an instance (G, k) of INDEPENDENT-SET that has an independent set of size k iff Φ is satisfiable
- Construction
  - G contains 3 vertices for each clause, one for each literal.
  - Connect 3 literals in a clause in a triangle.
  - Connect literal to each of its negations.

G



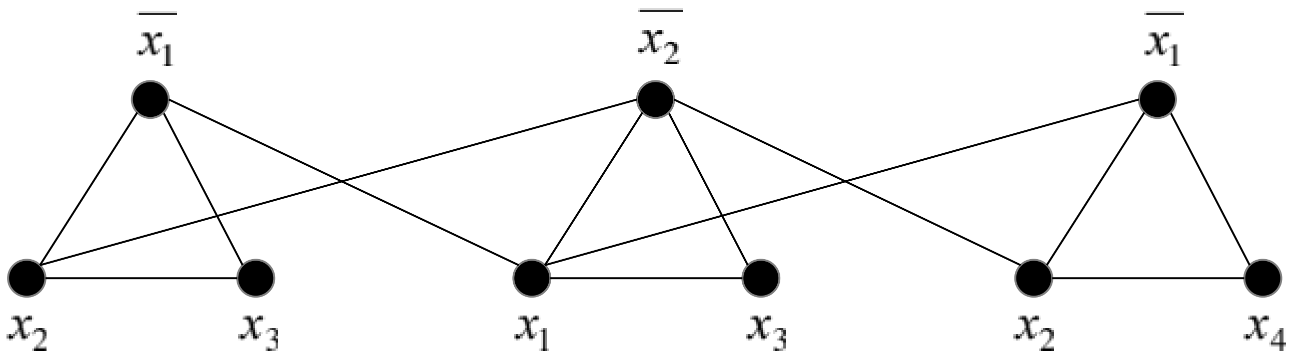
k = 3

$$\Phi = \left(\overline{x_1} \vee x_2 \vee x_3\right) \wedge \left(x_1 \vee \overline{x_2} \vee x_3\right) \wedge \left(\overline{x_1} \vee x_2 \vee x_4\right)$$

3-CNF $\leq_p$ INDEPENDENT-SET

- Claim: G contains independent set of size k = $|\Phi|$ iff Φ is satisfiable
- Proof: "⇒" Let S be independent set of size k
  - S must contain exactly one vertex in each triangle
  - Set these literals to true
  - Truth assignment is consistent and all clauses are satisfied

G



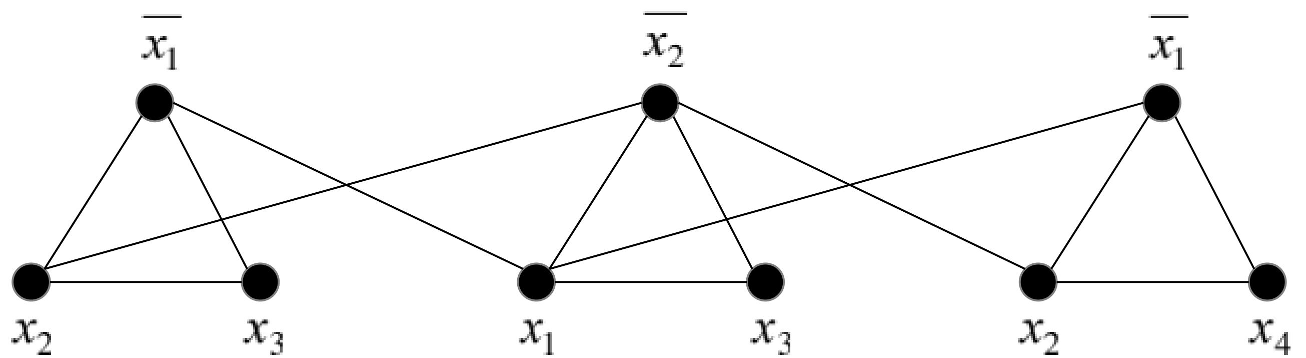
k = 3

$$\Phi = \left(\overline{x_1} \vee x_2 \vee x_3\right) \wedge \left(x_1 \vee \overline{x_2} \vee x_3\right) \wedge \left(\overline{x_1} \vee x_2 \vee x_4\right)$$

3-CNF $\leq_p$ INDEPENDENT-SET

- Claim: G contains independent set of size k = $|\Phi|$ iff Φ is satisfiable
- Proof: "$\Longleftarrow$"
  - Each triangle has a literal that evaluates to 1
  - This is an independent set S of size k
    - If there would be an edge between vertices in S, they would have to conflict

G



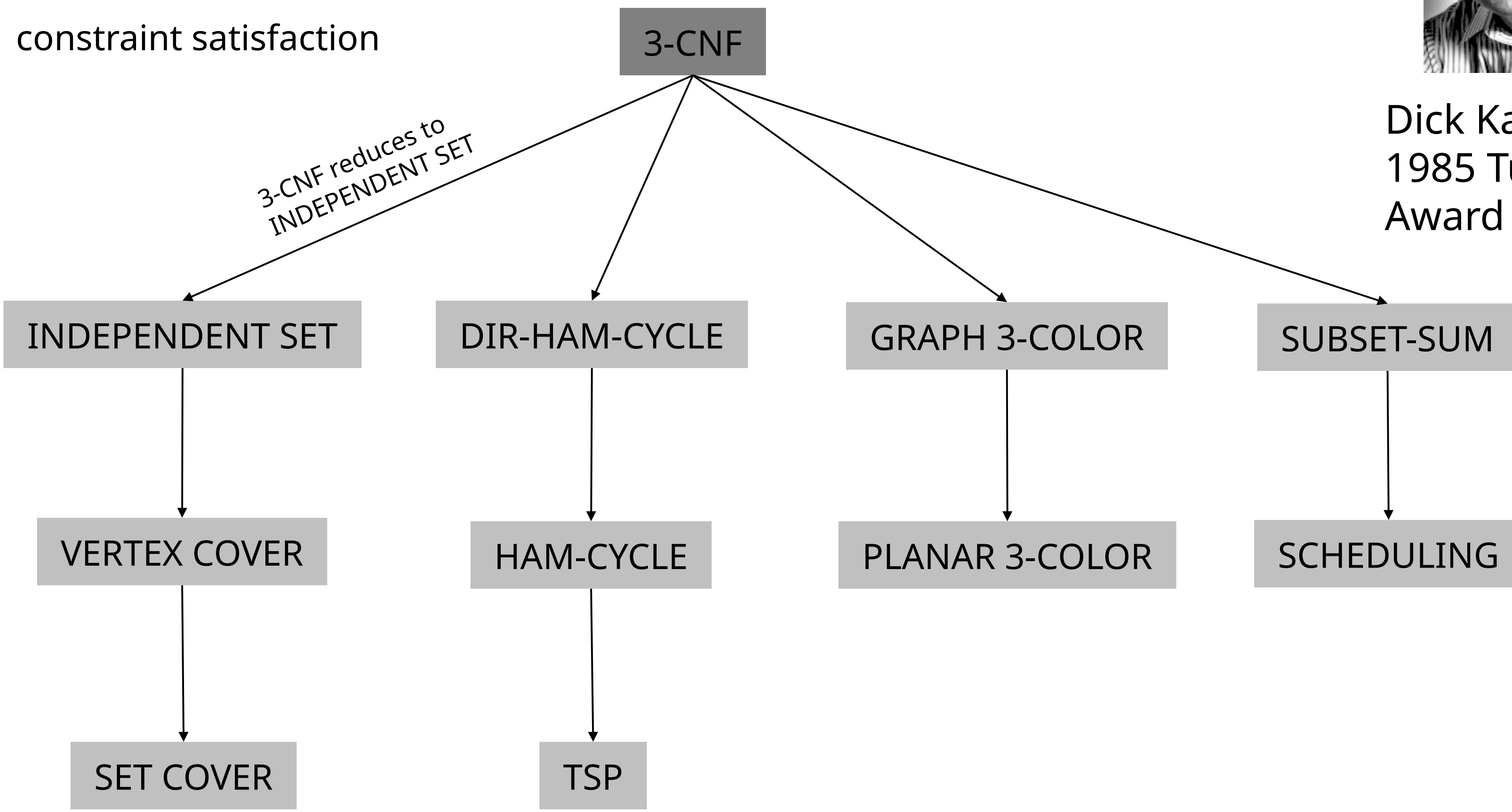
k = 3

$$\Phi = \left(\overline{x_1} \vee x_2 \vee x_3\right) \wedge \left(x_1 \vee \overline{x_2} \vee x_3\right) \wedge \left(\overline{x_1} \vee x_2 \vee x_4\right)$$

Polynomial-Time Reductions

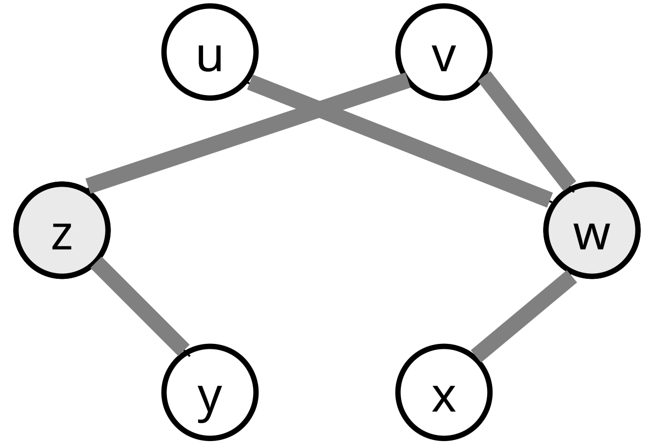
Dick Karp (1972)
1985 Turing Award

constraint satisfaction



packing and covering | sequencing | partitioning | numerical

Vertex Cover



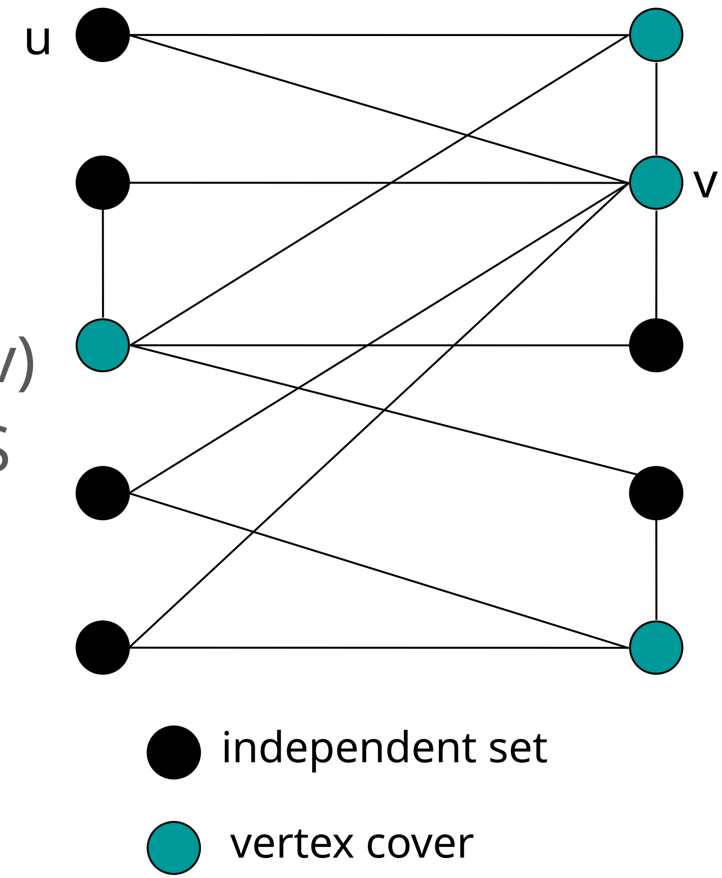
- G = (V, E), undirected graph
- **Vertex cover** = a subset V’ ⊆ V which covers all the edges
  - if (u, v) ∈ E then u ∈ V’ or v ∈ V’ or both.
- **Size** of a vertex cover = number of vertices in it

Problem:

- Find a vertex cover of minimum size
- Does graph G have a vertex cover of size k?

INDEPENDENT-SET $\leq_p$ VERTEX-COVER

- We show S is an independent set iff V ⟺ S is a vertex cover

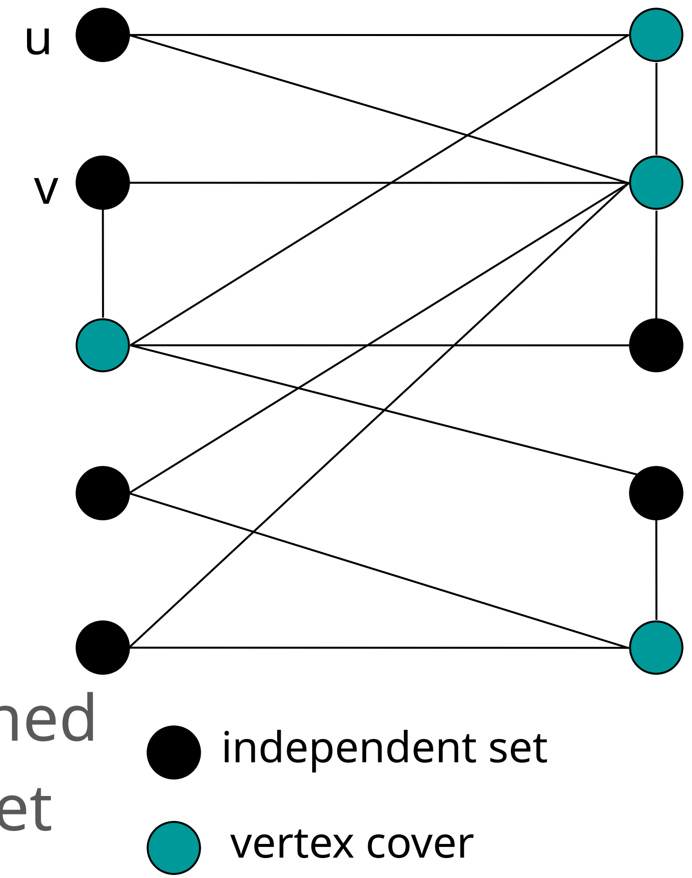


Proof "⇒"

- Let S be any independent set
- Consider an arbitrary edge (u, v)
- S independent ⇒ u ∉ S or v ∉ S
  ⇒ u ∈ V - S or v ∈ V - S
- Thus, V - S covers (u, v)

INDEPENDENT-SET $\leq_p$ VERTEX-COVER

- We show S is an independent set iff V $\Longleftrightarrow$ S is a vertex cover



Proof "$\Longleftarrow$"

- Let V - S be any vertex cover
- Consider two nodes u ∈ S and v ∈ S
- Observe that (u, v) ∉ E since V - S is a vertex cover
- Thus, no two nodes in S are joined by an edge $\Rightarrow$ S independent set

Set Cover

- Given a set U of elements, a collection $S_1, S_2, \ldots, S_m$ of subsets of U, and an integer k, does there exist a collection of ≤ k of these sets whose union is equal to U?

- Example

U = { 1, 2, 3, 4, 5, 6, 7 }

k = 2

$S_1 = \{3, 7\}$ $S_4 = \{2, 4\}$

$S_2 = \{3, 4, 5, 6\}$ $S_5 = \{5\}$

$S_3 = \{1\}$ $S_6 = \{1, 2, 6, 7\}$

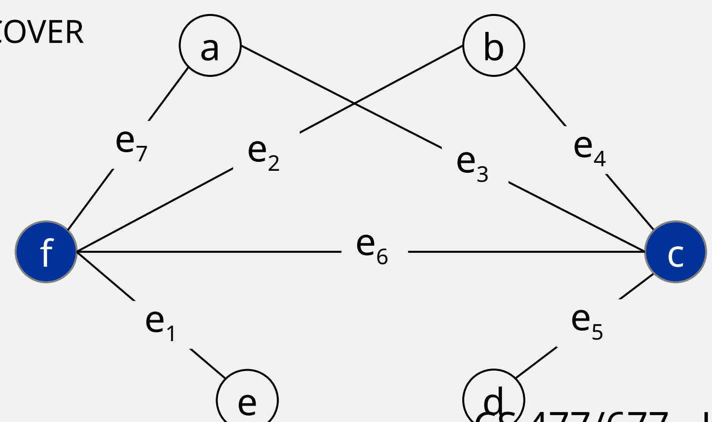
Set Cover

- Given a set U of elements, a collection $S_1, S_2, \ldots, S_m$ of subsets of U, and an integer k, does there exist a collection of ≤ k of these sets whose union is equal to U?
- Sample application
  - m available pieces of software
  - Set U of n capabilities that the system should have
  - The i-th piece of software provides the set $S_i \subseteq U$ of capabilities
  - Goal: achieve all n capabilities using fewest pieces of software

VERTEX-COVER $\leq_p$ SET-COVER

- Given a VERTEX-COVER instance $G = (V, E)$, k, we construct a set cover instance whose size equals the size of the vertex cover instance
- Construction
  - Create SET-COVER instance
    - $k = k$, $U = E$, $S_v = \{e \in E : e \text{ incident to } v\}$
  - Set-cover of size $\leq k$ iff vertex cover of size $\leq k$

VERTEX COVER



k = 2

SET COVER

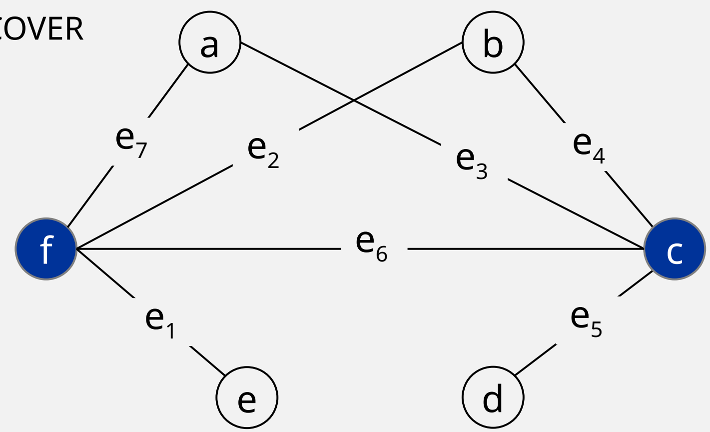
$U = \{1, 2, 3, 4, 5, 6, 7\}$
$k = 2$

$S_a = \{3, 7\}$ $S_b = \{2, 4\}$
$S_c = \{3, 4, 5, 6\}$ $S_d = \{5\}$
$S_e = \{1\}$ $S_f = \{1, 2, 6, 7\}$

VERTEX-COVER $\leq_p$ SET-COVER

- Set-cover of size $\leq k$ iff vertex cover of size $\leq k$
- Proof "⇒" (S_{i1}, …., S_{il} are $l \leq k$ sets that cover U)
  - Every edge in G is incident on one of the vertices $i_1,\ldots, i_l$, so $\{i_1,\ldots, i_l\}$ is a vertex cover of size $l \leq k$
- Proof "⟸" $\{i_1,\ldots, i_l\}$ is a vertex cover of size $l \leq k$
  - Then, the sets S_{i1}, …., S_{il} cover U

VERTEX COVER



k = 2

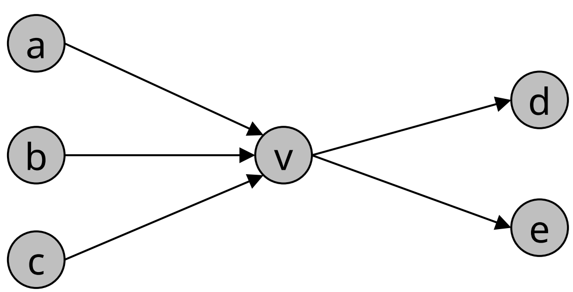
SET COVER

$U = \{ 1, 2, 3, 4, 5, 6, 7 \}$
$k = 2$

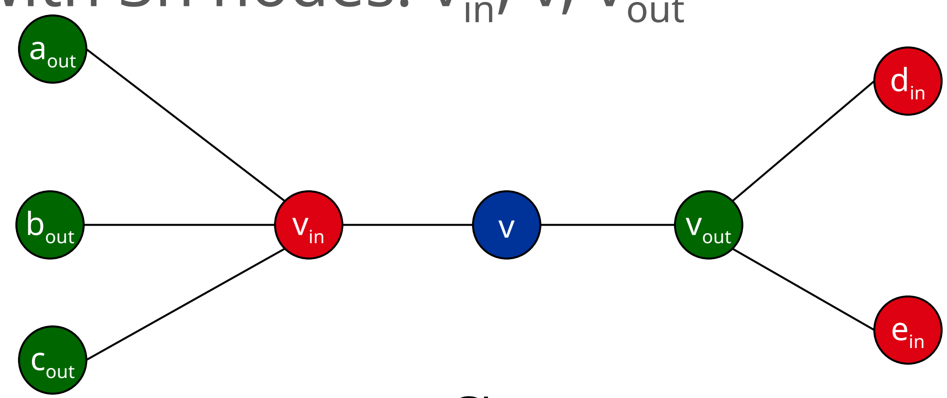
$S_a = \{3, 7\}$	$S_b = \{2, 4\}$
$S_c = \{3, 4, 5, 6\}$	$S_d = \{5\}$
$S_e = \{1\}$	$S_f = \{1, 2, 6, 7\}$

Hamiltonian Cycle

- Given an undirected graph G = (V, E), does there exists a simple directed cycle Γ that contains every node in V?
- Claim: DIR-HAM-CYCLE $\leq_P$ HAM-CYCLE
- Construction
  - Given a directed graph G = (V, E), construct an undirected graph G' with 3n nodes: v_{in}, v, v_{out}



G

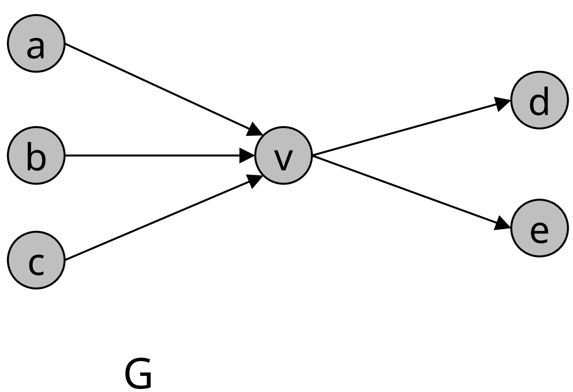


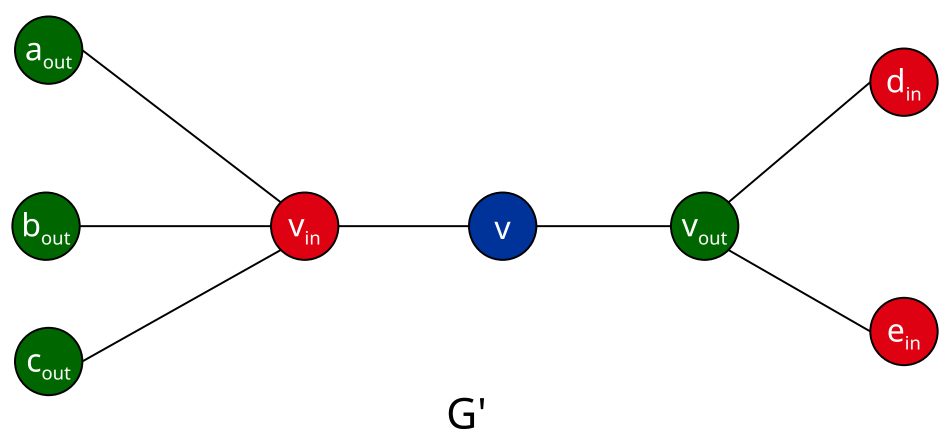
G'

DIR-HAM-CYCLE $\leq_p$ HAM-CYCLE

- Claim: G has a Hamiltonian cycle iff G' does.

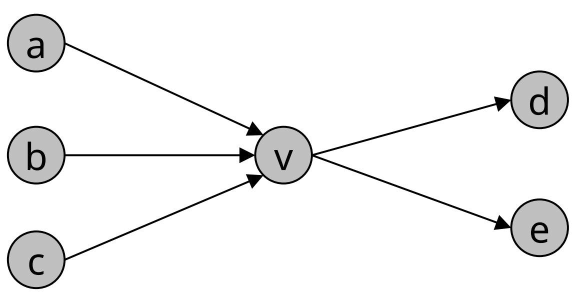
- Proof: "⇒"
  - Suppose G has a directed Hamiltonian cycle Γ
  - Then G' has an undirected Hamiltonian cycle (same order)



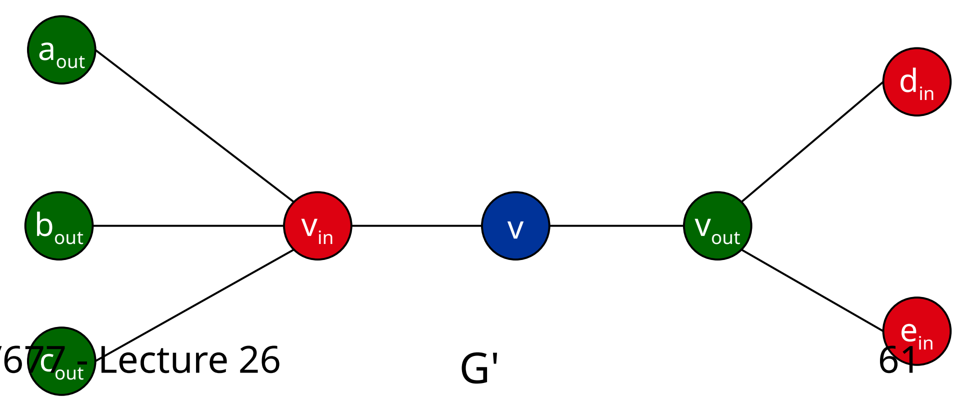


DIR-HAM-CYCLE $\leq_P$ HAM-CYCLE

- Claim: G has a Hamiltonian cycle iff G' does.
- Proof: "$\Longleftarrow$"
  - Suppose G' has an undirected Hamiltonian cycle Γ'
  - Γ' must visit nodes in G' using one of following two orders:
    - …, B, G, R, B, G, R, B, G, R, B, …
    - …, B, R, G, B, R, G, B, R, G, B, …
  - Blue nodes in Γ' make up directed Hamiltonian cycle Γ in G, or reverse of one



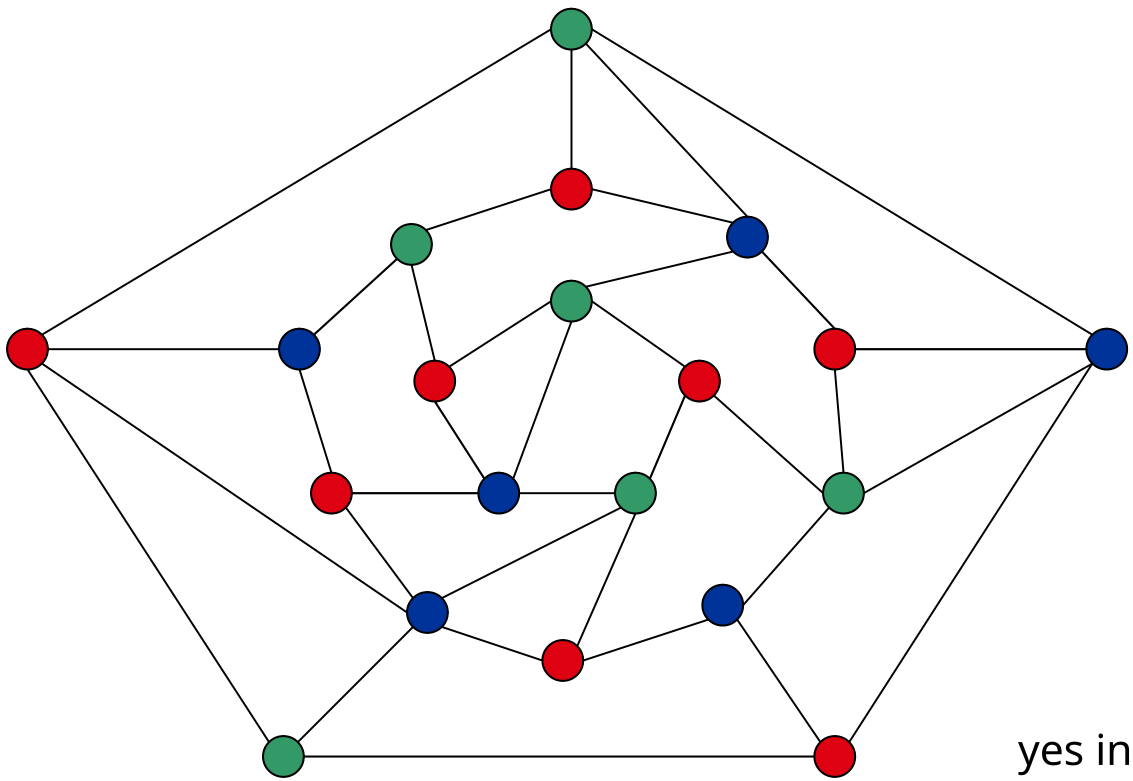
G



G'

3-Colorability

- Given an undirected graph G does there exists a way to color the nodes red, green, and blue so that no adjacent nodes have the same color?



yes instance

Register Allocation

- Register allocation
  - Assign program variables to machine register so that no more than k registers are used and no two program variables that are needed at the same time are assigned to the same register
- Interference graph
  - Nodes are program variables names, edge between u and v if there exists an operation where both u and v are "live" at the same time.
- Observation [Chaitin 1982]
  - Can solve register allocation problem iff interference graph is k-colorable
- Fact
  - 3-COLOR $\leq_P$ k-REGISTER-ALLOCATION for any constant $k \geq 3$

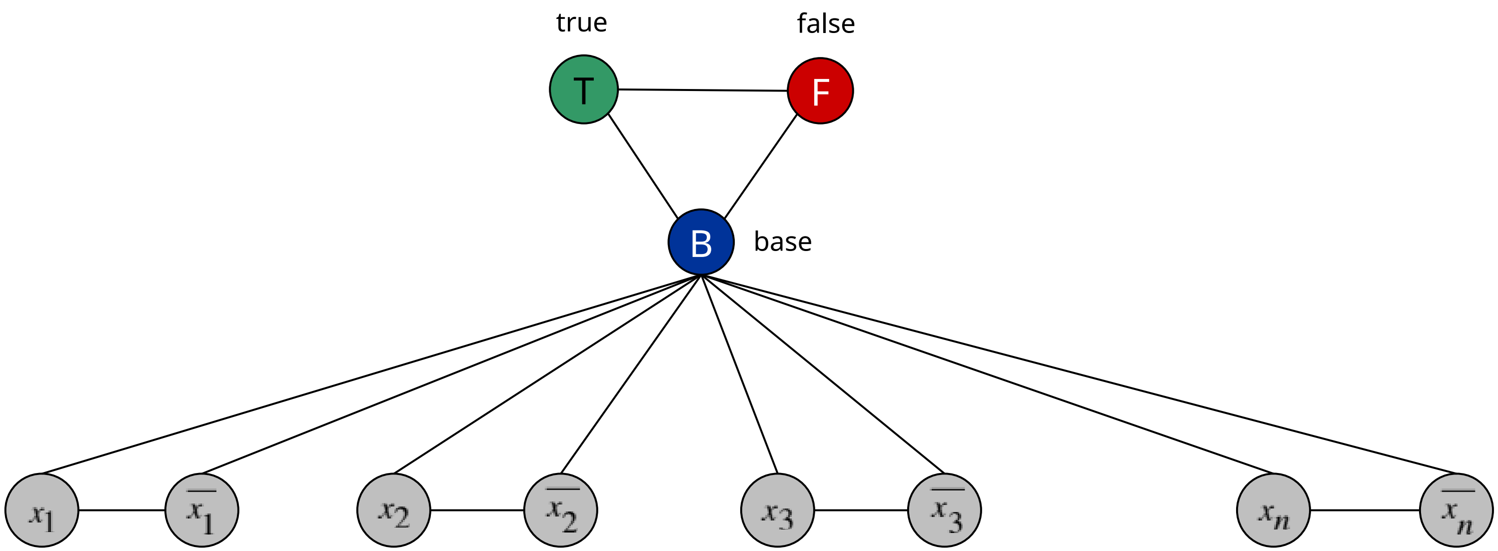
3-CNF $\leq_p$ 3-COLOR

- Given 3-CNF instance Φ, we construct an instance of 3-COLOR that is 3-colorable iff Φ is satisfiable

- Construction
  - For each literal, create a node
  - Create 3 new nodes T, F, B; connect them in a triangle, and connect each literal to B
  - Connect each literal to its negation
  - For each clause, add a 6-node subgraph

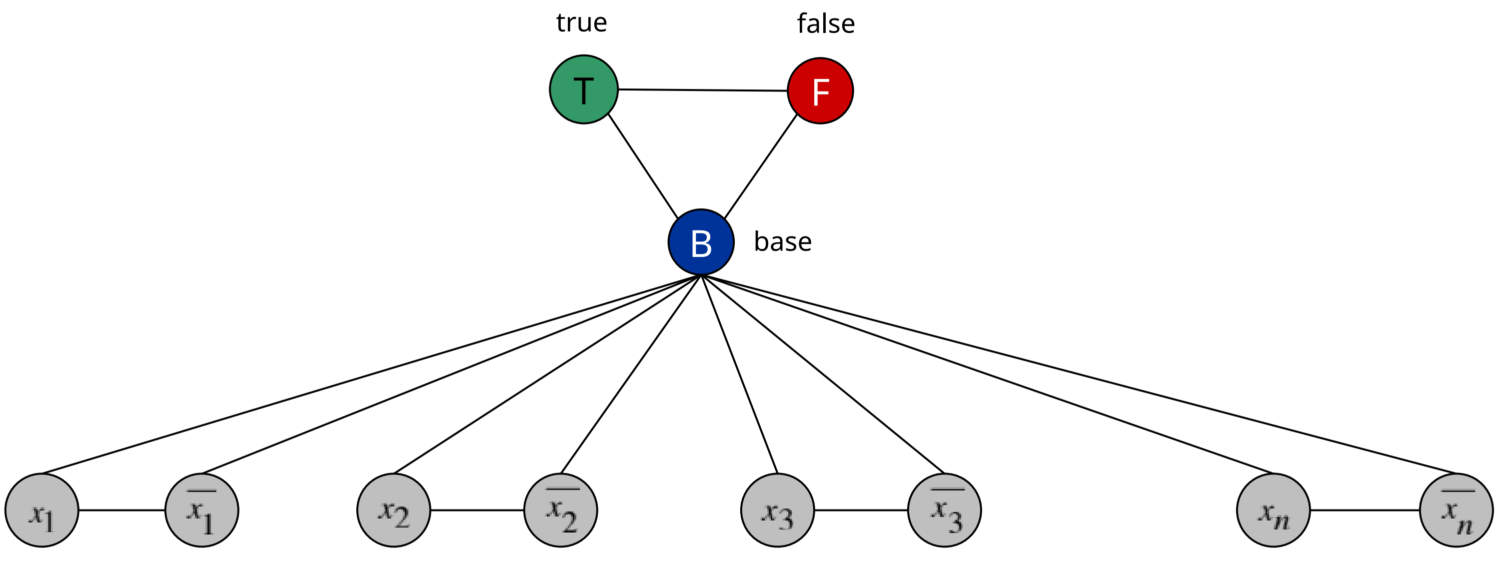
3-CNF $\leq_p$ 3-COLOR

- For each literal, create a node
- Create 3 new nodes T, F, B; connect them in a triangle, and connect each literal to B
- Connect each literal to its negation



3-CNF $\leq_p$ 3-COLOR

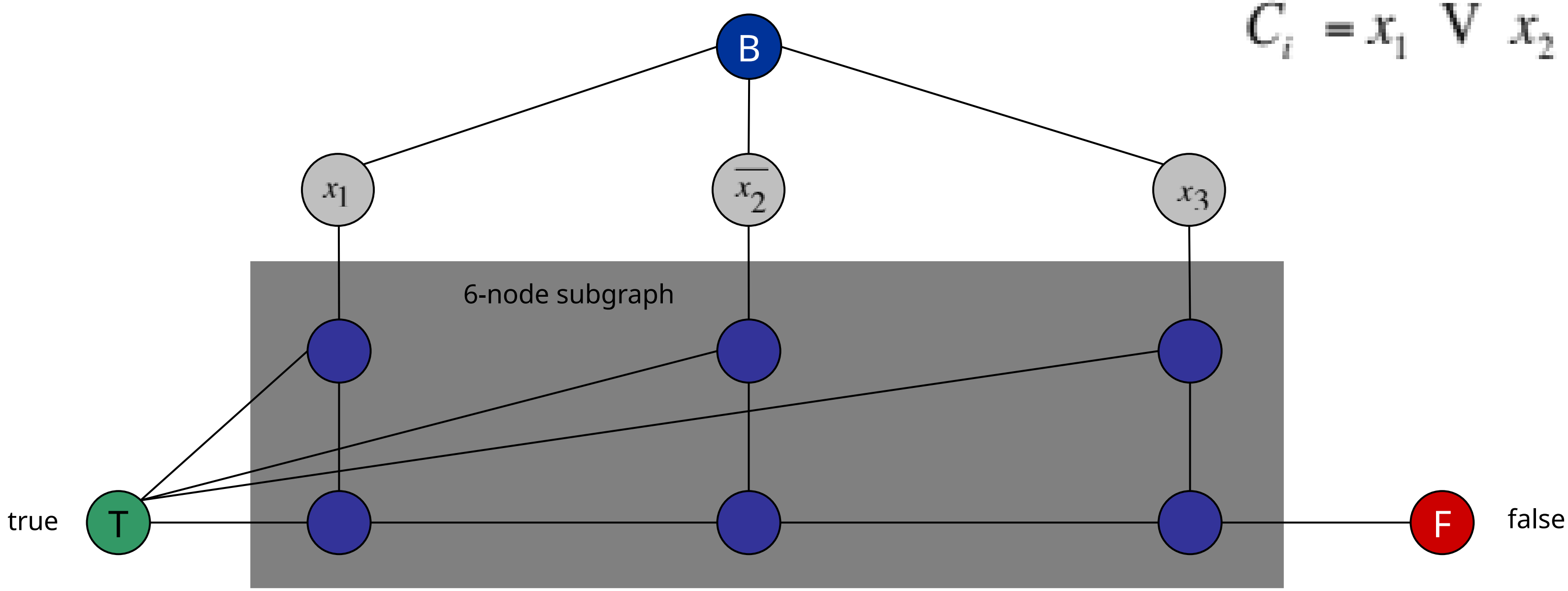
- Any 3-coloring implicitly determines a truth assignment for variables in 3-CNF
  - Nodes T, F, B must get different colors
  - For x_i and not-x_i, one will take T color one F color



3-CNF $\leq_p$ 3-COLOR

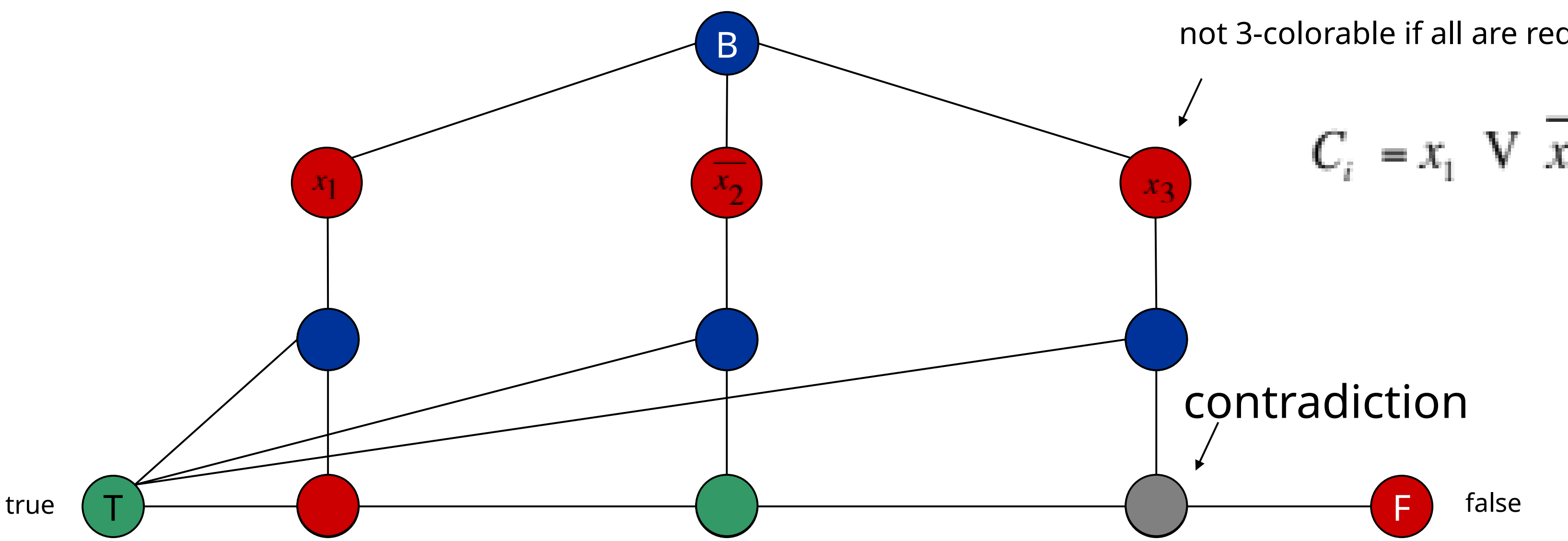
- Must ensure that only satisfying assignments can result in 3-coloring of the full graph
  - For each clause, add a 6-node subgraph

$C_i = x_1 \lor \overline{x_2} \lor x_3$



3-CNF $\leq_p$ 3-COLOR

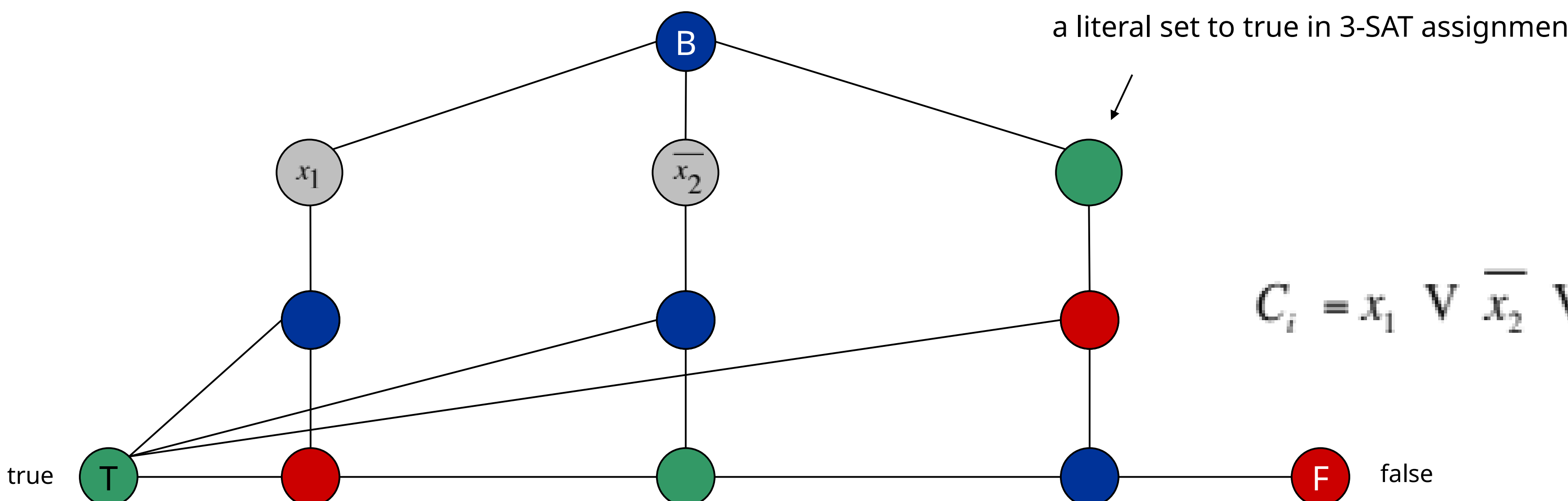
- Proof "⇒" Suppose graph is 3-colorable
  - Proof by contradiction: assume that all three literals get a False color



$$C_i = x_1 \vee \overline{x_2} \vee x_3$$

3-CNF $\leq_p$ 3-COLOR

- Proof "$\Longleftarrow$" Suppose 3-CNF formula Φ is satisfiable
  - Color all true literals T
  - Color node below green node F, and node below B
  - Color remaining middle row nodes B
  - Color remaining bottom nodes T or F as forced



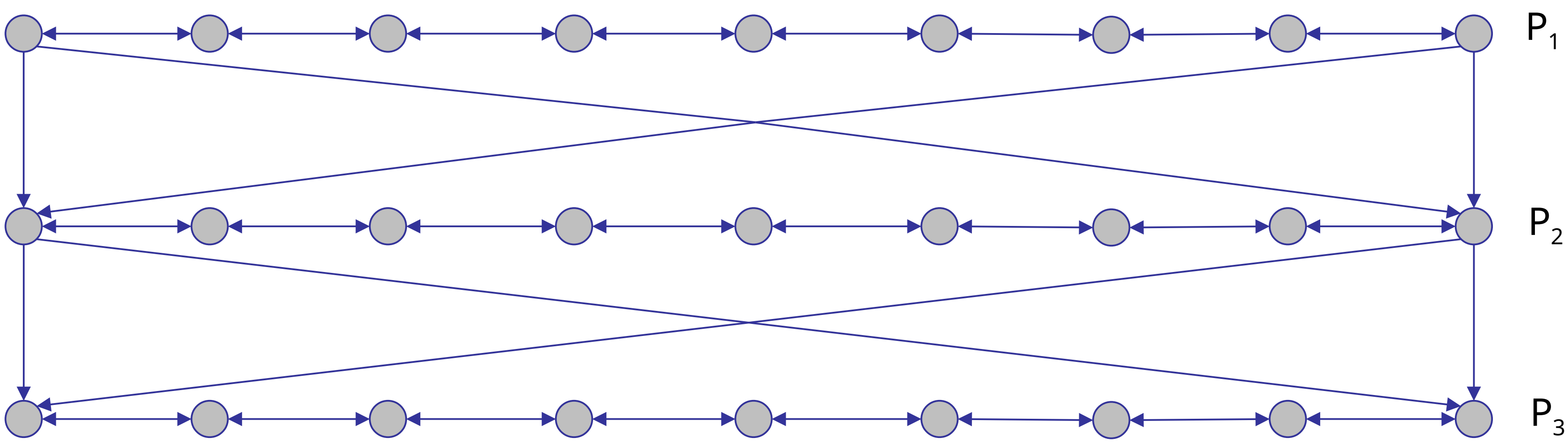
$$C_i = x_1 \vee \overline{x_2} \vee x_3$$

Directed Hamiltonian Cycle

- Given a digraph G = (V, E), does there exists a simple directed cycle Γ that contains every node in V?
- Idea:
  - Given an instance Φ of 3-CNF, we construct an instance of DIR-HAM-CYCLE that has a Hamiltonian cycle iff Φ is satisfiable
- Construction
  - Create a graph that has 2^n Hamiltonian cycles which correspond in a natural way to 2^n possible truth assignments

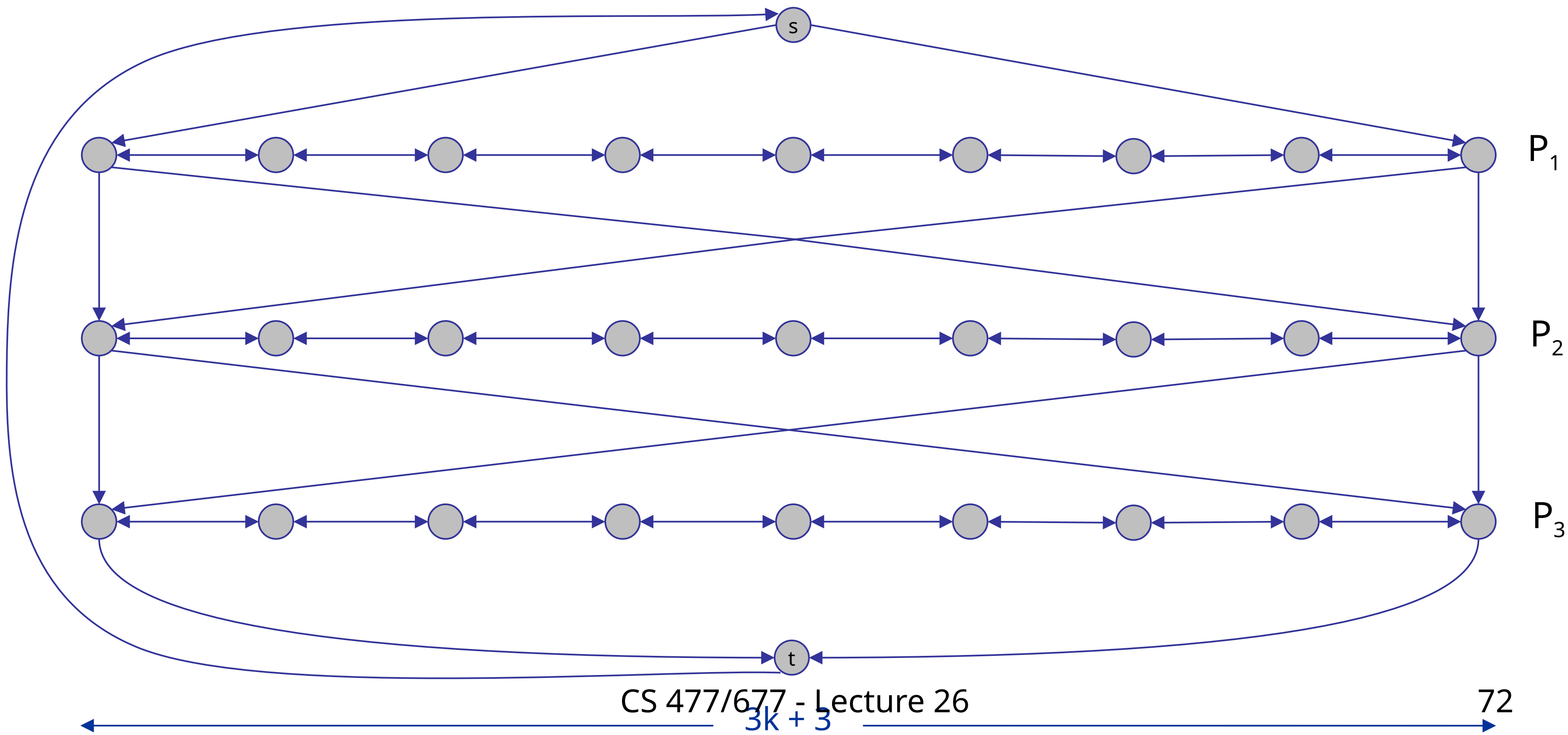
3-CNF $\leq_p$ DIR-HAM-CYCLE

- Construction: given 3-CNF instance Φ with n variables x_i and k clauses $C_1, \ldots, C_k$
  - Construct n paths $P_1, \ldots, P_n$, with P_i containing $v_{i1}, v_{i2}, \ldots, v_{ib}$
  - There are edges between adjacent vertices on path in each direction
  - Hook the paths together with edges



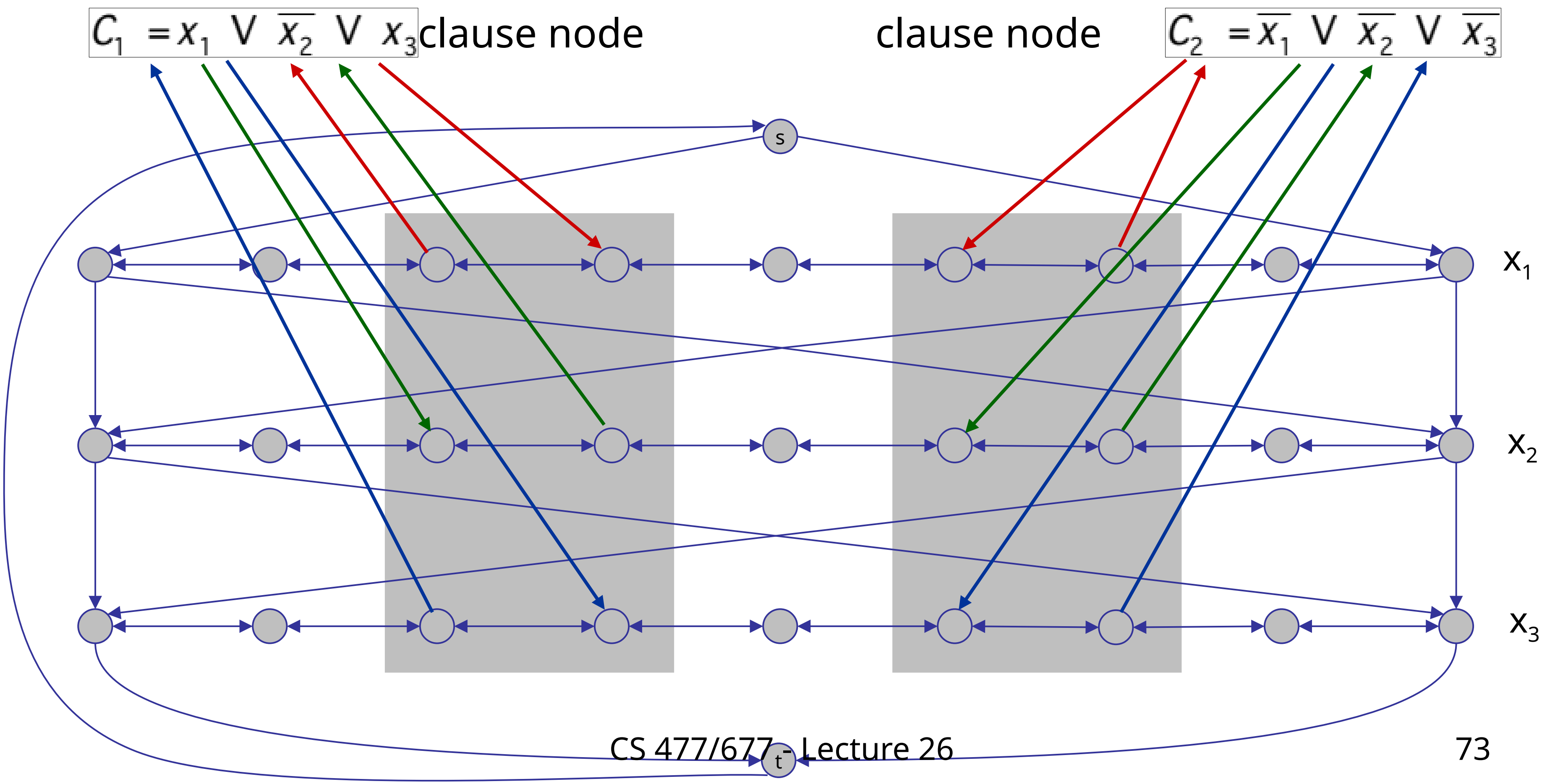
3-CNF $\leq_p$ DIR-HAM-CYCLE

- Construction (continued)
  - Add two vertices s and t and connect them with edges
  - Add edge from t to s
  - Intuition: cycle traverses path P_i from left to right $\Leftrightarrow$ set $x_i = 1$



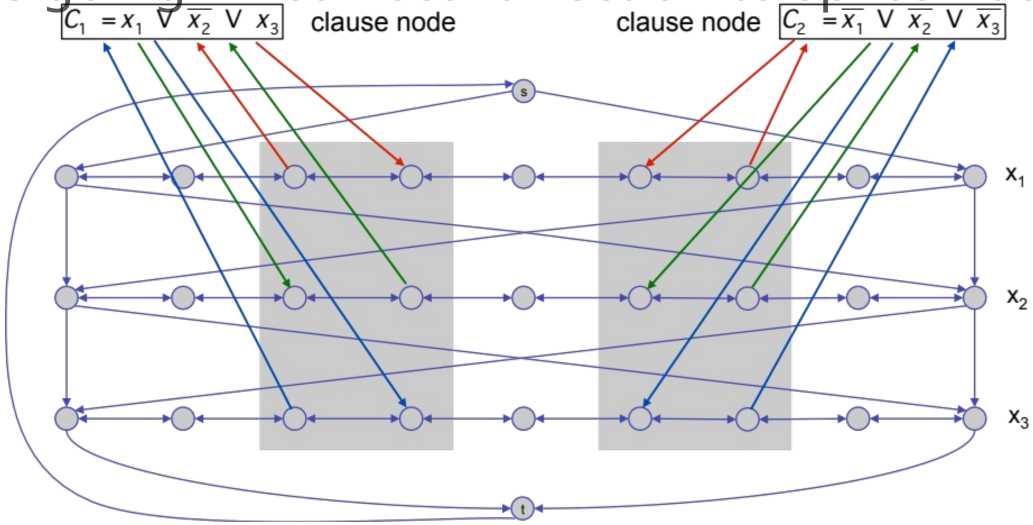
3-CNF $\leq_p$ DIR-HAM-CYCLE

- Construction (continued)
  - For each clause: add a node and 6 edges



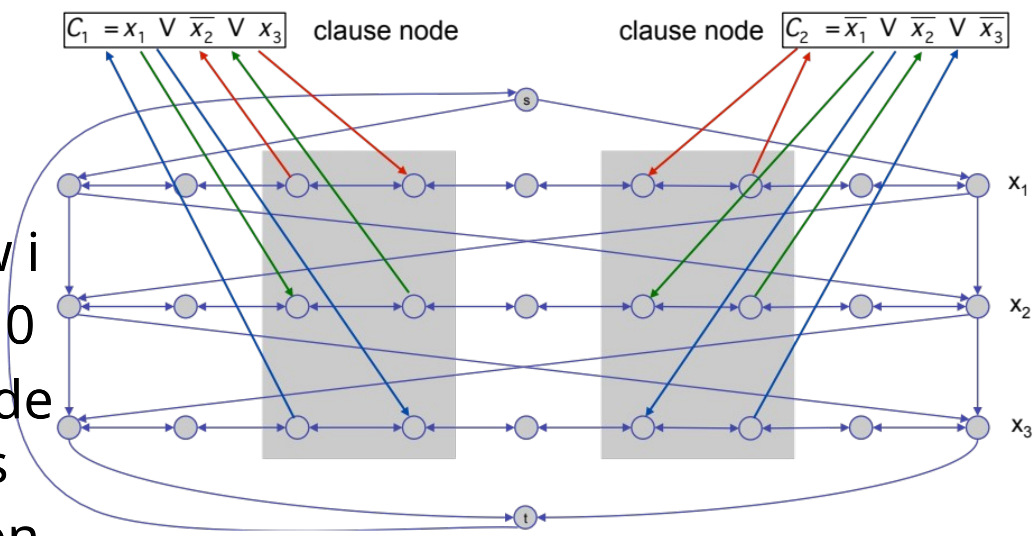
3-CNF $\leq_p$ DIR-HAM-CYCLE

- Claim: Φ is satisfiable iff G has a Hamiltonian cycle
- Proof "⇒" Suppose 3-CNF has satisfying assignment x*
  - Then, define Hamiltonian cycle in G as follows:
    - If $x_i^* = 1$, traverse row i from left to right
    - If $x_i^* = 0$, traverse row i from right to left
    - For each clause C_j, there will be at least one row i in which we are going in "correct" direction to splice node C_j into tour



3-CNF $\leq_p$ DIR-HAM-CYCLE

- Claim: Φ is satisfiable iff G has a Hamiltonian cycle
- Proof "$\Longleftarrow$" Suppose G has a Hamiltonian cycle Γ
  - If Γ enters clause node C_j, it must depart on mate edge
    - Nodes before and after C_j are connected by an edge e in G
    - Removing C_j from cycle, replace it with edge e $\Rightarrow$ Hamiltonian cycle on G - { C_j }
  - Continuing in this way, $\Rightarrow$ Hamiltonian cycle Γ' in G - { C_1 , C_2 , . . . , C_k }
  - Set $x_i^* = 1$ iff Γ' traverses row i left to right, otherwise set to 0
  - Since Γ visits each clause node C_j, at least one of the paths is traversed in "correct" direction, and each clause is satisfied

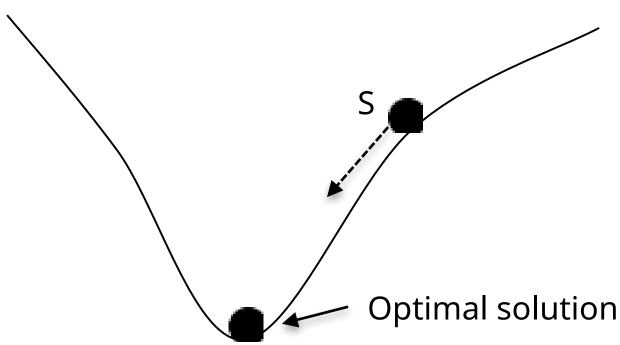


Optional, not required for final exam

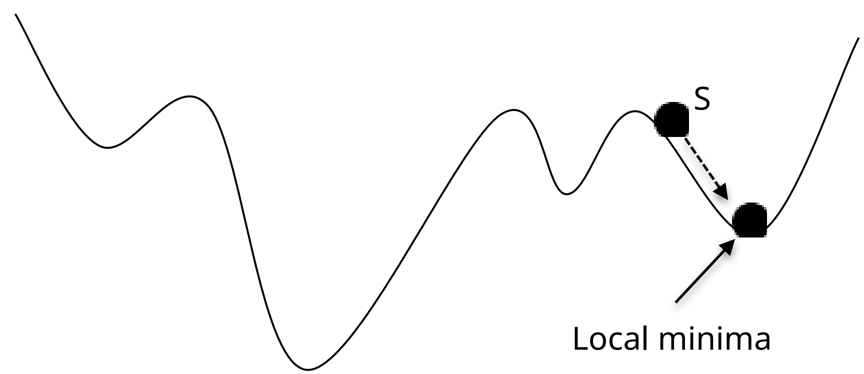
ADDITIONAL APPROXIMATION ALGORITHMS

Local Search (Hill Climbing, Gradient Descent)

- Explore the space of possible solutions, moving from a current solution to a "nearby" one
  1. Let S denote current solution
  2. If there is a neighbor S' of S with strictly lower cost, replace S with the neighbor whose cost is as small as possible
  3. Otherwise, terminate the algorithm



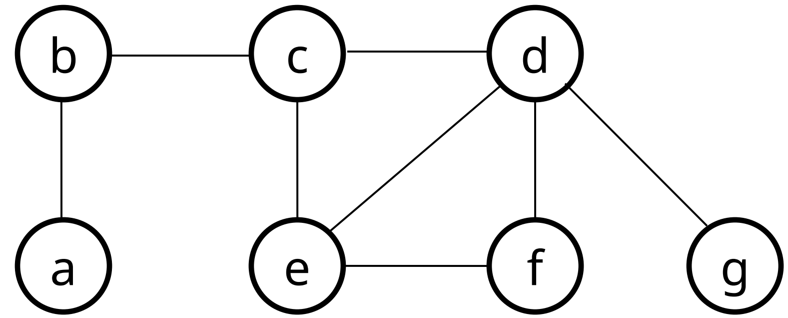
A funnel



A jagged funnel

The Vertex-Cover Problem

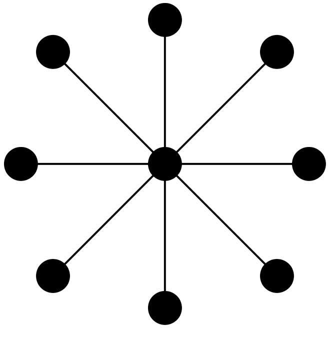
- Vertex cover of G = (V, E), undirected graph
  - A subset V' ⊆ V that covers all the edges in G



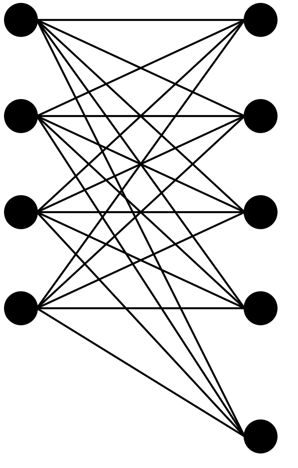
- **Hill climbing (gradient descent) idea**:
  - Start with a solution S = V
  - If there is a neighbor S' that is a vertex cover and has lower cardinality, replace S with S'.
  - Algorithm ends after at most n steps (each update decreases the size of the cover by one)

Gradient Descent: Vertex Cover

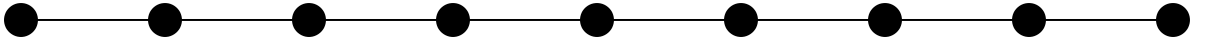
- Local optimum. No neighbor is strictly better.



optimum = center node only
local optimum = all other nodes



optimum = all nodes on left side
local optimum = all nodes on right side



optimum = even nodes
local optimum = omit every third node

The Set Covering Problem

- Finite set X
- Family F of subsets of X: $F = \{S_1, S_2, \ldots, S_n\}$

$$X = \bigcup_{S \in F} S$$

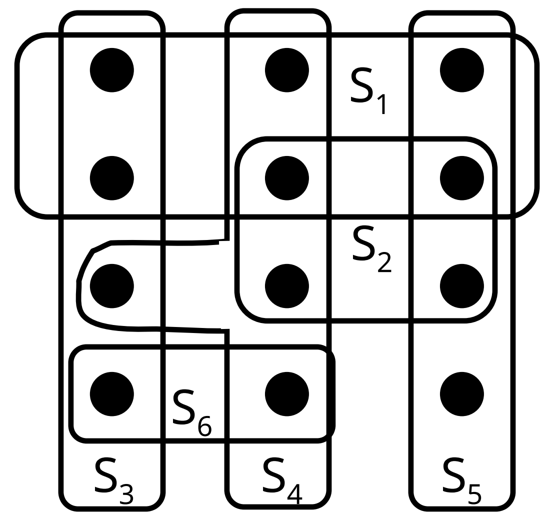
- Find a minimum-size subset $C \subseteq F$ that covers all the elements in X
- Decision: given a number k find if there exist k sets $S_{i1}, S_{i2}, \ldots, S_{ik}$ such that:

$$S_{i1} \cup S_{i2} \cup \ldots \cup S_{ik} = X$$

Greedy Set Covering

Idea:

- At each step pick a set S that covers the greatest number of remaining elements



Optimal: C = $\{S_3, S_4, S_5\}$

GREEDY-SET-COVER(X, F)

1. $U \leftarrow X$
2. $C \leftarrow \emptyset$
3. **while** $U \neq \emptyset$
4. **do** select an $S \in F$ that maximizes $|S \cap U|$
5. $U \leftarrow U - S$
6. $C \leftarrow C \cup \{S\}$
7. **return** C