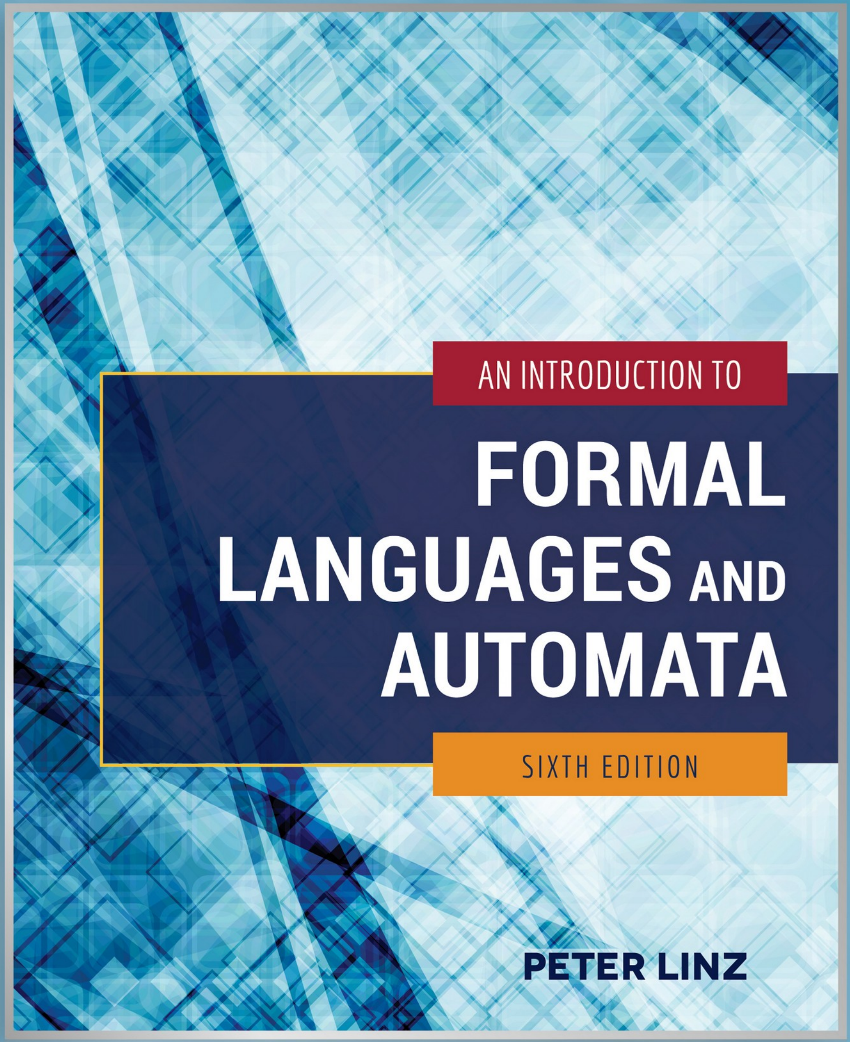
Chapter 8

PROPERTIES OF CONTEXT-FREE LANGUAGES



Learning Objectives

At the conclusion of the chapter, the student will be able to:

- Apply the pumping lemma to show that a language is not context-free
- State the closure properties applicable to context-free languages
- Prove that context-free languages are closed under union, concatenation, and star-closure
- Prove that context-free languages are not closed under either intersection or complementation
- Describe a membership algorithm for context-free languages
- Describe an algorithm to determine if a context-free language is empty
- Describe an algorithm to determine if a context-free language is infinite

A Pumping Lemma for Context-Free Languages

- Theorem 8.1: Given an infinite context-free language L, every sufficiently long string *w* in L can be broken into four parts *uvxyz* such that
  - $|vy| \geq 1$
  - $|vxy| \leq m$ (where m is an arbitrary integer $\leq |w|$)
  - An arbitrary, but equal number of repetitions of v and y yields another string in L
- The "pumped" string consists of two separate parts (v and y) and can occur anywhere in the string
- The pumping lemma can be used to show that, by contradiction, a certain language is not context-free

An Illustration of the Pumping Lemma for Context-Free Languages

As shown in Figure 8.1, the pumping lemma for context-free languages can be illustrated by sketching a general derivation tree that shows a decomposition of the string into the required components

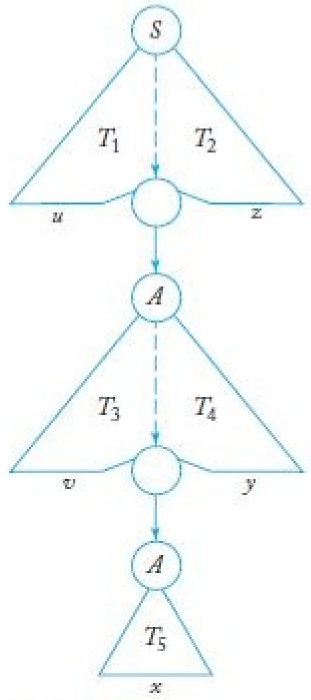


FIGURE 8.1

Closure Properties for Context-Free Languages

- Theorem 8.3 states that if L_1 and L_2 are context-free languages, so are the languages that result from the following operations:
  - $L_1 \cup L_2$
  - $L_1 \cap L_2$
  - L_1^*
- In other words, the family of regular languages is closed under union, intersection, and star-closure.
- To prove these properties, we assume the existence of two context-free grammars G_1 and G_2 that generate the respective languages

Proof of Closure under Union

- Assume that L_1 and L_2 are generated by the context-free grammars $G_1 = (V_1, T_1, S_1, P_1)$ and $G_2 = (V_2, T_2, S_2, P_2)$
- Without loss of generality, assume that the sets V_1 and V_2 are disjoint
- Create a new variable S_3 which is not in $V_1 \cup V_2$
- Construct a new grammar $G_3 = (V_3, T_3, S_3, P_3)$ so that
  - $V_3 = V_1 \cup V_2 \cup \{ S_3 \}$
  - $T_3 = T_1 \cup T_2$
  - $P_3 = P_1 \cup P_2$
- Add to P_3 a production that allows the new start symbol to derive either of the start symbols for L_1 and L_2

  $S_3 \to S_1 \mid S_2$

Proof of Closure under Concatenation

- Assume that L_1 and L_2 are generated by the context-free grammars $G_1 = (V_1, T_1, S_1, P_1)$ and $G_2 = (V_2, T_2, S_2, P_2)$
- Without loss of generality, assume that the sets V_1 and V_2 are disjoint
- Create a new variable S_4 which is not in $V_1 \cup V_2$
- Construct a new grammar $G_4 = (V_4, T_4, S_4, P_4)$ so that
  - $V_4 = V_1 \cup V_2 \cup \{ S_4 \}$
  - $T_4 = T_1 \cup T_2$
  - $P_4 = P_1 \cup P_2$
- Add to P_4 a production that allows the new start symbol to derive the concatenation of the start symbols for L_1 and L_2

  $S_4 \rightarrow S_1S_2$
- Clearly, G_4 is context-free and generates the concatenation of L_1 and L_2, thus completing the proof

Proof of Closure under Star-Closure

- Assume that L_1 is generated by the context-free grammars $G_1 = (V_1, T_1, S_1, P_1)$
- Create a new variable S_5 which is not in V_1
- Construct a new grammar $G_5 = (V_5, T_5, S_5, P_5)$ so that
  - $V_5 = V_1 \cup \{ S_5 \}$
  - $T_5 = T_1$
  - $P_5 = P_1$
- Add to P_5 a production that allows the new start symbol S_5 to derive the repetition of the start symbol for L_1 any number of times

  $S_5 \rightarrow S_1S_5 \mid \lambda$
- Clearly, G_5 is context-free and generates the star-closure of L_1, thus completing the proof

No Closure under Intersection

- Unlike regular languages, the intersection of two context-free languages L_1 and L_2 does not necessarily produce a context-free language
- As a counterexample, consider the context-free languages

$$L_1 = \{ a^n b^n c^m : n \geq 0, m \geq 0 \}$$

$$L_2 = \{ a^n b^m c^m : n \geq 0, m \geq 0 \}$$

- However, the intersection L_1 and L_2 is the language

$$L_3 = \{ a^n b^n c^n : n \geq 0 \}$$

- L_3 can be shown not be context-free by applying the pumping lemma for context-free languages

No Closure under Complementation

- The complement of a context-free language L_1 does not necessarily produce a context-free language
- The proof is by contradiction: given two context-free languages L_1 and L_2, assume that their complements are also context-free
- By Theorem 8.3, the union of the complements must also produce a context-free language L_3
- Using our assumption, the complement of L_3 is also context-free
- However, using the set identity below, we conclude that the complement of L_3 is the intersection of L_1 and L_2, which has been shown not t contradicting our assumption.

$$L_1 \cap L_2 = \overline{\overline{L_1} \cup \overline{L_2}}.$$

Elementary Questions about Context-Free Languages

- Given a context-free language L and an arbitrary string w, is there an algorithm to determine whether or not w is in L?
- Given a context-free language L, is there an algorithm to determine if L is empty?
- Given a context-free language L, is there an algorithm to determine if L is infinite?
- Given two context-free grammars G_1 and G_2, is there an algorithm to determine if $L(G_1) = L(G_2)$?

A Membership Algorithm for Context-Free Languages

- The combination of Theorems 5.2 and 6.5 confirms the existence of a membership algorithm for context-free languages
- By Theorem 5.2, exhaustive parsing is guaranteed to give the correct result for any context-free grammar that contains neither λ-productions nor unit-productions
- By Theorem 6.5, such a grammar can always be produced if the language does not include λ
- Alternatively, a npda to accept the language can be constructed as established by Theorem 7.1

Determining Whether a Context-Free Language is Empty

- Theorem 8.6 confirms the existence of an algorithm to determine if a context-free language L(G) is empty
- For simplicity, assume that λ is not in L(G)
- Apply the algorithm for removing useless symbols and productions
- If the start symbol is found to be useless, then L(G) is empty; otherwise, L(G) contains at least one string

Determining Whether a Context-Free Language is Infinite

- Theorem 8.7 confirms the existence of an algorithm to determine if a context-free language L(G) is infinite
- Apply the algorithms for removing λ-productions, unit-productions, and useless productions
- If G has a variable A for which there is a derivation that allows A to produce a sentential form xAy, then L(G) is infinite
- Otherwise, L(G) is finite
- Can be implemented by building a dependency graph which contains an edge from A to B for every rule of the form $A \rightarrow xBy$

Determining Whether Two Context-Free Languages are Equal

- Given two context-free grammars G_1 and G_2, is there an algorithm to determine if $L(G_1) = L(G_2)$?
- If the languages are finite, the answer can be found by performing a string-by-string comparison
- However, for general context-free languages, <u>no algorithm exists to determine equality</u>